Closed Geodesics on Compact Hyperbolic Surfaces

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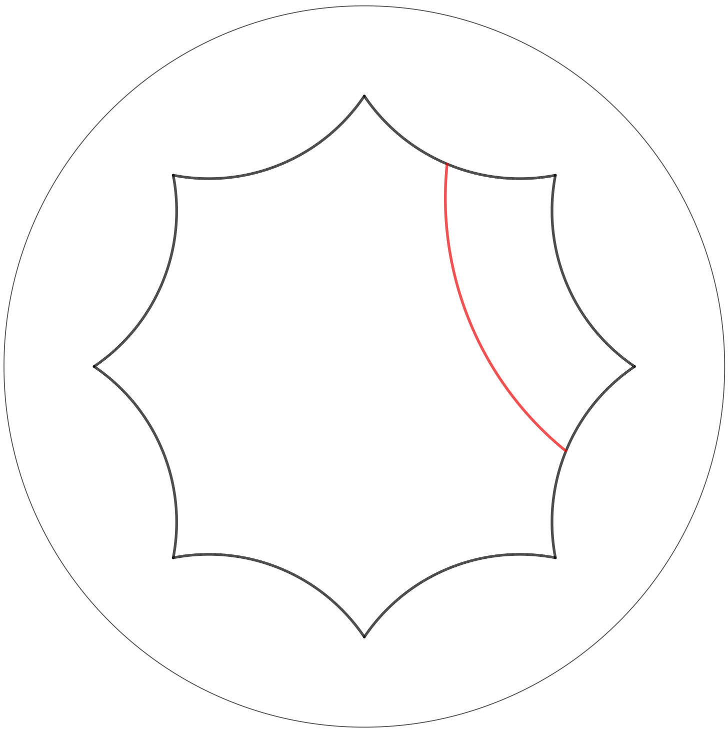

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Preface

This thesis is concerned with counting curves on surfaces. More precisely, we are interested in closed geodesics on hyperbolic surfaces. A rigorous definition of the terms *closed geodesic* and *hyperbolic surface* will be given in section 1, yet let me briefly explain these expressions for the unfamiliar reader.

A *geodesic* is essentially a straight line on a general surface or manifold. More formally, a geodesic is a locally length minimizing curve. For example, picture the two-dimensional torus, which we view as a square in the plane where the opposite edges are associated. Figure 1 shows three curves on the torus connecting two points. The left image is not a geodesic, whereas the middle and the right ones are indeed geodesics.

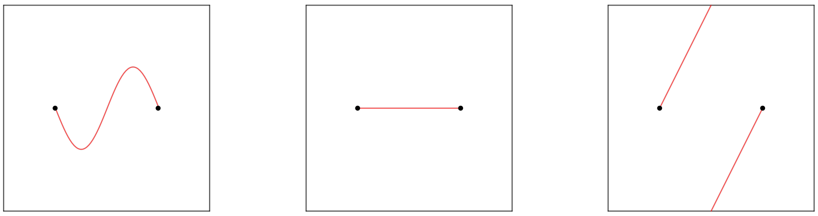

Figure 1: Curves on the torus

A geodesic is called *closed*, if it is a loop as in Figure 2.

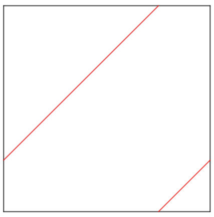

Figure 2: Closed geodesic on the torus

In order to describe hyperbolic surfaces, we first discuss the two dimensional hyperbolic disk. A model of the two dimensional hyperbolic disk is given by the unit disk $\mathbb{D} = \{x + iy \in \mathbb{C} : x^2 + y^2 < 1\}$. However, we don't measure distances in the standard euclidean way. Instead we consider a distance function such that the following curves in $\mathbb{D}$ are geodesics:

(i) Lines through the center of the disk.

(ii) Circle arcs that are orthogonal to the boundary of the disk.

Figure 3 depicts the unit disk together with some geodesics.

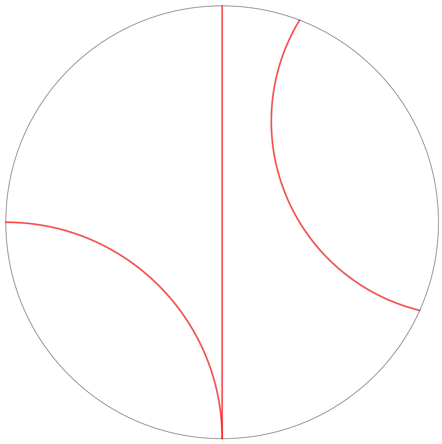

Figure 3: Geodesics on $\mathbb{H}$

In this setting, *hyperbolic surfaces* are roughly speaking geodesic polygons in $\mathbb{D}$, where the edges are glued together in a suitable manner. Figure 4 shows a geodesic octagon in $\mathbb{D}$ where the edges are glued together in such a manner that the polygon forms a surface of genus 2. We also show in Figure 4 a closed geodesic on the surface. Note further that the hyperbolic surface depicted in Figure 4 is compact.

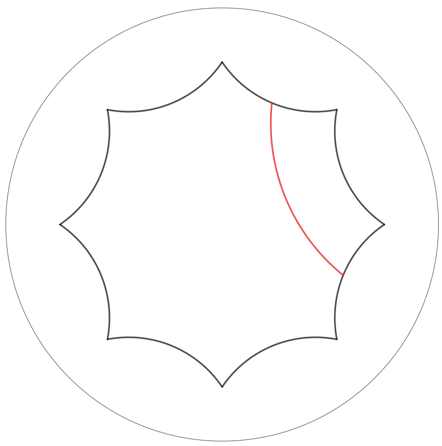

Figure 4: A hyperbolic surface with a closed geodesic

Compact hyperbolic surfaces have the interesting property that there are only finitely many closed geodesics of length less than a fixed number L. Hence we can ask how this number behaves as L tends to infinity. The main objective of this thesis is to show that the number of closed geodesics of length less than L behaves asymptotically like

$$\frac{e^L}{L},$$

as L tends to infinity. Note that if we replace L by $\log(n)$, we get

$$\frac{n}{\log(n)}.$$

The reader interested in prime numbers will recognize this growth rate. Namely, the prime number theorem states that the counting function

$$\pi(n) := |\{\text{prime numbers less than } n\}|$$

also also behaves asymptotically as $\frac{n}{\log(n)}$. This is not just a coincidence. We can translate the notion of a prime number to the set of closed geodesics on a hyperbolic surface. This is done as follows.

A closed geodesic is called *prime*, if it is not the iterate of another closed geodesic. If we think of prime geodesics as prime numbers, then the set of closed geodesics corresponds to the set of prime powers in $\mathbb{Z}$. Thus it suffices to understand the asymptotics of the number of prime geodesics in order to answer questions concerning the asymptotic behavior of the number of closed geodesics. In analogy to the prime number theorem, an estimate for the number of prime geodesics is given by the prime geodesic theorem, which implies the above mentioned rougher estimate for the number of closed geodesics.

In order to prove the prime geodesic theorem, we will derive the Selberg trace formula for compact hyperbolic surfaces, which connects eigenvalues of the Laplace operator to the length of closed geodesics. We will briefly explain how the trace is to be understood in this context. Recall from linear algebra that the trace of a symmetric matrix can be expressed as the sum of its eigenvalues. This notion generalizes to a certain class of linear operators on Hilbert spaces. It turns out that for self-adjoint operators the trace is again equal to the sum of its eigenvalues. The Selberg trace formula is derived by relating the trace of certain operators to expressions of geometric meaning such as the length of closed geodesics.

As a prerequisite to understand the Selberg trace formula, we need to study the Laplace operator on hyperbolic surfaces. We will prove that on compact hyperbolic surfaces there is a countable number of eigenvalues and we discuss that the sum of the inverse square of these converges.

We will proceed as follows. Section 1 introduces hyperbolic geometry, hyperbolic surfaces and geodesics on these spaces. Then we study the Laplace operator in section 2 and discuss the necessary statements concerning eigenvalues. The thesis culminates with section 3, where we prove the Selberg trace formula and the prime geodesic theorem.

This thesis should be readable by anybody acquainted with the curriculum of the first two years of undergraduate study at ETH, i.e. we assume familiarity with basic notions from analysis, algebra, topology and measure theory. The necessary prerequisites concerning topological groups and functional analysis are contained in the appendix. We refrain from using the language of Riemannian geometry, except in a few informal discussions in section 1.

Finally, I would like to express my sincere gratitude to Professor Marc Burger for many helpful discussions and for providing me with his notes on the Selberg trace formula. Furthermore, I would like to thank Professor Dietmar Salamon for helping me find an appropriate topic for my bachelor thesis. Lastly, I thank my colleagues Nicholas Dykeman, Dominique Heyn and Johannes Ladwig for proofreading my thesis.

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Constantin Kogler

1 Hyperbolic Space and Geodesics

On the sphere and the torus there are an uncountable number of distinct closed geodesics. Furthermore, picturing compact surfaces in three dimensional euclidean space, it seems unexceptional that there are uncountably many closed geodesics. How is it then sensible to count the number of closed geodesics on surfaces?

Compact hyperbolic surfaces have the surprising property that every closed loop is freely homotopic to a unique closed geodesic. Note that this is false in the case of the sphere and the torus. A further mysterious characteristic of compact hyperbolic surfaces is that there are only finitely many closed geodesics with length less than a fixed number. Therefore we can study the asymptotic behavior of the number of closed geodesics of increasing length.

This section is concerned with describing geodesics on hyperbolic surfaces. In the first section we discern the geodesics on the upper half plane. In the subsequent section we descends to hyperbolic surfaces and prove the aforementioned statements. The main references for this section are [Bea91],[Bro16], and [EW11].

1.1 Geodesics on the Upper Half Plane

In contrast to the introduction of this thesis we study hyperbolic space now with another model, namely the upper half plane $\mathbb{H} = \{x + iy : x, y \in \mathbb{R} \text{ and } y > 0\}$. We consider $\mathbb{H}$ together with the Riemannian metric

$$\frac{1}{y^2}(dx^2 + dy^2).$$

A Riemannian metric defines the length of curves. In this case, for a smooth curve $\phi : I \to \mathbb{H}$, with I is some interval, the length of ϕ is defined as

$$L(\phi) := \int_I ||\phi'(t)||_{\phi(t)}\, dt := \int_I \frac{||\phi'(t)||}{\mathrm{Im}(\phi(t))}\, dt,$$

where $|| \circ ||$ denotes the euclidean norm of $\phi(t)$ viewed as an element of $\mathbb{R}^2$. This determines a metric on $\mathbb{H}$ via

$$d(z, w) = \inf_{\phi} L(\phi),$$

with ϕ varying over all smooth curves from $z \in \mathbb{H}$ to $w \in \mathbb{H}$.

The group

$$\mathrm{SL}_2(\mathbb{R}) := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2} : \det(g) = ad - bc = 1 \right\}$$

acts on $\mathbb{H}$ via Möbius transformations given for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ by

$$\mathbb{H} \to \mathbb{H}, \qquad z \mapsto g \circ z = gz = \frac{az + b}{cz + d}. \tag{1.1}$$

We mostly will use the notation gz instead of $g \circ z$. This action is well defined as the following equation shows:

$$\mathrm{Im}(gz) = \frac{\mathrm{Im}(z)}{|cz + d|^2}. \tag{1.2}$$

In order to prove (1.2), calculate for $z = x + iy$:

$$\begin{aligned}\operatorname{Im}(gz) = \operatorname{Im}\left(\frac{az+b}{cz+d}\right) &= \operatorname{Im}\left(\frac{az+b}{cz+d}\frac{c\overline{z}+d}{c\overline{z}+d}\right)\\ &= \operatorname{Im}\left(\frac{ac|z|^2+bd+adz+bc\overline{z}}{c^2|z|^2+d^2+2dcx}\right)\\ &= \operatorname{Im}\left(\frac{iy(ad-bc)}{c^2|z|^2+d^2+2dcx}\right)\\ &= \frac{\operatorname{Im}(z)}{|cz+d|^2}.\end{aligned}$$

Furthermore, one also checks that this is indeed a group action, i.e. for all $g_1, g_2 \in \mathrm{SL}_2(\mathbb{R})$ and all $z \in \mathbb{H}$,

$$g_2(g_1 z) = (g_2 g_1)z.$$

Proposition 1.1. *The group action of* $\mathrm{SL}_2(\mathbb{R})$ *on* $\mathbb{H}$ *defined by equation (1.1) is isometric and transitive. Furthermore, the stabilizer at* $i \in \mathbb{H}$ *is* $\mathrm{SO}_2(\mathbb{R})$.

Proof. To prove the first claim, we need to show that for all $z, w \in \mathbb{H}$ and all $g \in \mathrm{SL}_2(\mathbb{R})$ we have

$$d(gz, gw) = d(z, w).$$

It suffices to check that g preserves the length of curves. So we need to show that for any curve $\phi : [0,1] \to \mathbb{H}$ and any $g \in \mathrm{SL}_2(\mathbb{R})$,

$$L(g \circ \phi) = L(\phi).$$

With the help of (1.2), we compute

$$\begin{aligned}L(g\circ\phi) &= \int_0^1 \frac{||\frac{d}{dt}(g\circ\phi)(t)||}{\operatorname{Im}(g\circ\phi(t))}\,dt\\ &= \int_0^1 \frac{||g'(\phi(t))\cdot\phi'(t)||}{\operatorname{Im}(g\circ\phi(t))}\,dt\\ &= \int_0^1 \frac{||(\frac{1}{c\phi(t)+d})^2\cdot\phi'(t)||}{\frac{\operatorname{Im}(\phi(t))}{|c\phi(t)+d|^2}}\,dt\\ &= \int_0^1 \frac{||\phi'(t)||}{\operatorname{Im}(\phi(t))}\,dt = L(\phi).\end{aligned}$$

Next, we show that the action is transitive. It suffices to check that for all $z = x + iy \in \mathbb{H}$ there is some element $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ such that $g \circ i = z$. The following element of $\mathrm{SL}_2(\mathbb{R})$ satisfies this:

$$\begin{pmatrix}\sqrt{y} & \frac{x}{\sqrt{y}}\\ 0 & \frac{1}{\sqrt{y}}\end{pmatrix} \circ i = \frac{\sqrt{y}i + \frac{x}{\sqrt{y}}}{0i + \frac{1}{\sqrt{y}}} = x + iy = z.$$

Denote by $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ an element of the stabilizer at i. We hence have

$$ai + b = -c + di.$$

So $a = d$ and $b = -c$. Thus $a^2 + c^2 = 1$ because $g = \left(\begin{smallmatrix} a & -c \\ c & a \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, implying the existence of a $\theta \in \mathbb{R}$ with

$$g = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in \mathrm{SO}_2(\mathbb{R}).$$

By this, one easily concludes that the stabilizer is $\mathrm{SO}_2(\mathbb{R})$. □

We denote by

$$\mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R})/\{\pm I_2\}, \qquad \mathrm{PSO}_2(\mathbb{R}) := \mathrm{SO}_2(\mathbb{R})/\{\pm I_2\}.$$

Corollary 1.2. *There is an identification*

$$\mathbb{H} \cong \mathrm{PSL}_2(\mathbb{R})/\mathrm{PSO}_2(\mathbb{R}).$$

Proof. By Proposition 1.1 we conclude that

$$\mathbb{H} \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}).$$

Since $I \circ z = (-I) \circ z$ for all $z \in \mathbb{H}$, the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$ defined by (1.1) is well defined. As in Proposition 1.1 one shows that this action is isometric and transitive with the stabilizer at $i \in \mathbb{H}$ being $\mathrm{PSO}_2(\mathbb{R})$. □

We introduce the notion of an orthogonal circle. The aim of the next few pages is to show that orthogonal circles are precisely global geodesics on $\mathbb{H}$. Figure 5 shows some orthogonal circles on $\mathbb{H}$.

Definition 1.3. An *orthogonal circle* is either a line parallel to the imaginary axis in $\mathbb{H}$ or the part of a circle with center on the real axis which is contained in $\mathbb{H}$.

For constants $C_1, C_2, C_3 \in \mathbb{R}$ we fix the notation

$$P(C_1, C_2, C_3) := \{z \in \mathbb{H} : C_1|z|^2 + C_2(z + \overline{z}) + C_3 = 0\}.$$

Lemma 1.4. *A subset $O \subset \mathbb{H}$ is an orthogonal circle if and only if there exists constants $C_1, C_2, C_3 \in \mathbb{R}$ with $C_2^2 > C_1C_3$ and not both of C_1 and C_2 being zero such that*

$$O = P(C_1, C_2, C_3).$$

Proof. Assume O is a circle of radius $r > 0$ and center $m \in \mathbb{R} \subset \mathbb{C}$. Then

$$O = \{z \in \mathbb{H} : |z - m|^2 = r^2\}.$$

Consequently, if $z \in O$, then

$$0 = (z - m)(\overline{z} - m) - r^2 = |z|^2 - m(z + \overline{z}) + m^2 - r^2.$$

So we set $C_1 = 1$, $C_2 = -m$ and $C_3 = m^2 - r^2$ so indeed $C_2^2 > C_1C_3$.

Now assume that O is a line parallel to the imaginary axis. Then we can write

$$O = \{z \in \mathbb{H} : (z + \overline{z}) = c\} \tag{1.3}$$

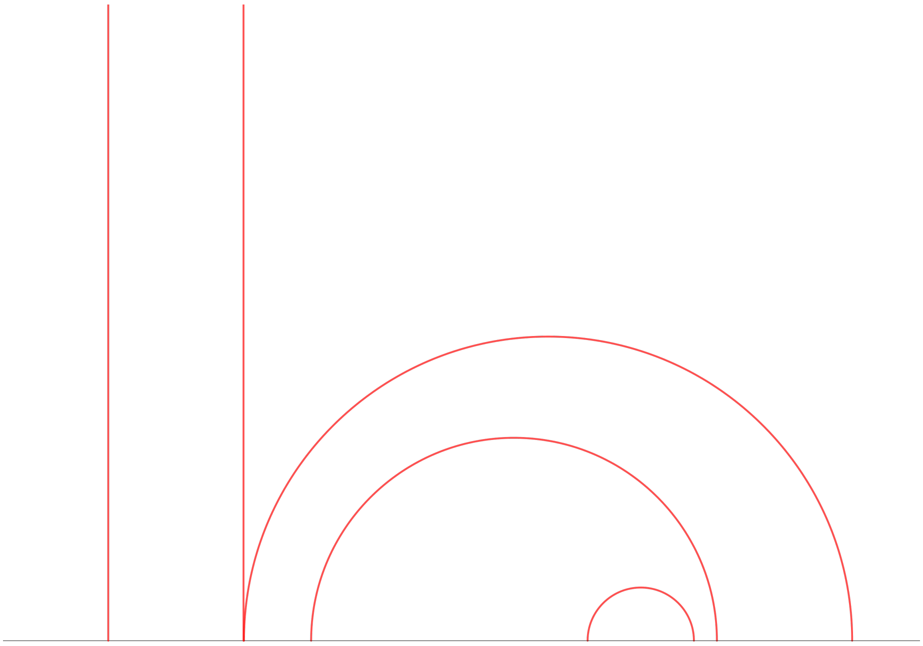

Figure 5: Orthogonal circles on $\mathbb{H}$

for some constant $c \in \mathbb{R}$. Hence $C_3 = -c$ and $C_2 = 1$ and $C_1 = 0$. So we also get $C_2^2 > C_1 C_3$.

Assume conversely that $C_1 \neq 0$. Then dividing by C_1 yields

$$P(C_1, C_2, C_3) = \{z \in \mathbb{H} \, : \, |z|^2 + \frac{C_2}{C_1}(z + \overline{z}) + \frac{C_3}{C_1} = 0\}$$

and with the notation of the first paragraph we conclude with $m = -\frac{C_2}{C_1}$ and $r^2 = \frac{C_3}{C_1} - m^2$ that $P(C_1, C_2, C_3)$ is a circle.

It remains to consider the case where $C_1 = 0$ and $C_2 \neq 0$. Then $P(C_1, C_2, C_3)$ is a set of the form 1.3, which is a line parallel to the imaginary axis. □

Lemma 1.5. *The group* $\mathrm{SL}_2(\mathbb{R})$ *is generated by matrices of the form*

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where $t \in \mathbb{R}$ *and* $s \in \mathbb{R}\backslash\{0\}$.

Proof. Each diagonal matrix in $\mathrm{SL}_2(\mathbb{R})$ is of the form $\left(\begin{smallmatrix} s & 0 \\ 0 & s^{-1} \end{smallmatrix}\right)$. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, observe that not both of a and b can be zero, hence we can assume, without loss of generality after possibly multiplying, with $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ that a is not zero. By multiplying with $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$, we get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & at + b \\ c & ct + d \end{pmatrix}.$$

So we can choose t such that $a + bt = 0$ and thus need to show that matrices of the form $\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right)$ can be generated by the above generators. In this case d cannot be zero. Observe

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -a \\ d & td - c \end{pmatrix}$$

Choose t such that $td - c = 0$ and multiply by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ to get a diagonal matrix. $\square$

Proposition 1.6. *The action of* $\mathrm{SL}_2(\mathbb{R})$ *on* $\mathbb{H}$ *defined by (1.1) maps orthogonal circles to orthogonal circles. Furthermore, for any two orthogonal circles* O_1 *and* O_2 *there is an element* $g \in \mathrm{SL}_2(\mathbb{R})$ *such that* $g \circ O_1 = O_2$.

Proof. It suffices to check the first statement on the generators of $\mathrm{SL}_2(\mathbb{R})$. This is clear for matrices of the form

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$$

where $t \in \mathbb{R}$ and $s \in \mathbb{R}\backslash\{0\}$. We now check the statement for the third type of generators. Any orthogonal circle O is given by

$$O = P(C_1, C_2, C_3) := \{z \in \mathbb{H} : C_1|z|^2 + C_2(z + \overline{z}) + C_3 = 0\},$$

with $C_2^2 > C_1C_3$. If we apply $g = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ we get

$$\begin{aligned} g \circ O = g \circ P(C_1, C_2, C_3) &= \left\{ z \in \mathbb{H} : \frac{C_1}{|z|^2} + \frac{C_2}{|z|^2}(z + \overline{z}) + C_3 = 0 \right\} \\ &= \left\{ z \in \mathbb{H} : C_1 + C_2(z + \overline{z}) + C_3|z|^2 = 0 \right\} \\ &= P(C_3, C_2, C_1). \end{aligned}$$

The set $P(C_3, C_2, C_1)$ is indeed an orthogonal circle by Lemma 1.4. So indeed the action by $\mathrm{SL}_2(\mathbb{R})$ maps orthogonal circles to orthogonal circles.

Since $\mathrm{SL}_2(\mathbb{R})$ is a group action, in order to prove the second statement it suffices to show that for every orthogonal circle O there is a element $g \in \mathrm{SL}_2(\mathbb{R})$ such that $g \circ O$ is the imaginary axis.

If O is parallel to the imaginary axis, then just use a translation of the form $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$. Now assume that O is a semi-circle with center on the real axis. Denote by r the left intersection point between the circle that contains O and the real axis and by p the point of the circle with largest imaginary coordinate. By action of a translation we can assume that $r = 0$. In order to map this circle to the imaginary axis, we want to find an element $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ such that $g \circ r = g \circ 0 = 0$ and $g \circ p = i$. If we find such an element then $g \circ O$ is the imaginary axis, since this is the only orthogonal circle that goes through 0 and i.

The condition $g \circ 0 = 0$ implies $b = 0$. If we write $p = x + iy$ the condition $g \circ p = i$ implies

$$ax + b = -cy \qquad \text{and} \qquad ay = cx + d.$$

Thus, since $b = 0$ we get that $ax = -cy$ and we set $d = ay - cx$. As $g \in \mathrm{SL}_2(\mathbb{R})$ we also need $ad - bc = 1$. With the above fixed variables we get

$$a^2y - acx = 1.$$

Plugging in $-c = \frac{ax}{y}$ (note that y is not zero since $t \in \mathbb{H}$) we get $a^2(y + \frac{x^2}{y}) = 1$. Then

$$a = \frac{1}{\sqrt{y + \frac{x^2}{y}}}.$$

So we get a matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a = \frac{1}{\sqrt{y+\frac{x^2}{y}}}$, $b = 0$, $c = -\frac{ax}{y}$ and $d = ay - cx$. □

Definition 1.7. A *geodesic between two points* $z, w \in \mathbb{H}$ is a smooth curve $\phi : [0,1] \to \mathbb{H}$ such that $\phi(0) = z$, $\phi(1) = w$ and

$$L(\phi) = d(z,w).$$

Remark. This definition coincides for $\mathbb{H}$ with the Riemannian definition of geodesics since, for the upper half plane with the hyperbolic Riemannian metric, the exponential map is a diffeomorphism.

Lemma 1.8. *Let $z = y_0 i$ and $w = y_1 i$ and $0 < y_0 < y_1$. Then*

$$d(z,w) = \log(y_1) - \log(y_0)$$

and the path

$$\phi : [0,1] \to \mathbb{H}, \qquad t \mapsto y_0 \left(\frac{y_1}{y_0}\right)^t i$$

is the unique geodesic from z_0 to z_1 with constant speed $\log(y_1) - \log(y_0)$.

Proof. We have that

$$||\dot{\phi}(s)||_{\phi(s)} = \frac{1}{y_0(\frac{y_1}{y_0})^t}||\dot{\phi}(t)|| = \frac{\log\left(\frac{y_1}{y_0}\right)}{y_0(\frac{y_1}{y_0})^t}||\phi(t)|| = \log(y_1) - \log(y_0).$$

So the curve is of constant speed $\log(y_1) - \log(y_0)$. Hence $d(z_0, z_1) \leq L(\phi) = \log(y_1) - \log(y_0)$. Consider now any other path $\eta : [0,1] \to \mathbb{H}$ joining z and w and write $\eta(s) = \eta_x(s) + i\eta_y(s)$ for $\eta_x(s), \eta_y(s) \in \mathbb{R}$. Then

$$\begin{aligned} L(\eta) &= \int_0^1 \frac{||\dot{\eta}(s)||}{\eta_y(s)}\, ds \\ &\geq \int_0^1 \frac{|\dot{\eta}_y(s)|}{\eta_y(s)}\, ds \\ &\geq \int_0^1 \frac{\dot{\eta}_y(s)}{\eta_y(s)}\, ds \\ &= \int_{y_0}^{y_1} \frac{1}{s}\, ds = \log(y_1) - \log(y_0) = L(\phi) \end{aligned}$$

So we proved $L(\phi) = d(z,w)$. Furthermore we have that $L(\eta) = L(\phi)$ if and only if $\dot{\eta}_x(t) = 0$ for all $t \in [0,1]$. Hence $\eta_x \equiv 0$ and thus $\dot{\eta}_y(t) \geq 0$ for all $t \in [0,1]$. So the above curve is the unique geodesic from z to w with constant speed $\log(y_1) - \log(y_0)$. □

Recall the definition of the hyperbolic cosine

$$\cosh(t) := \frac{1}{2}(e^t + e^{-t}).$$

Theorem 1.9. *For $z, w \in \mathbb{H}$ we have*

$$\cosh(d(z,w)) = 1 + \frac{|z-w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)}. \tag{1.4}$$

Proof. Observe that the left hand side is $\mathrm{SL}_2(\mathbb{R})$-invariant since the action is isometric. We will show that the right hand side is also invariant under $\mathrm{SL}_2(\mathbb{R})$. Observe for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$,

$$\begin{aligned} |cz+d| \cdot |cw+d| \cdot |gz - gw| &= |cz+d| \cdot |cw+d| \cdot \left| \frac{az+b}{cz+d} - \frac{aw+b}{cw+d} \right| \\ &= |(cw+d)(az+b) - (cz+d)(aw+b)| \\ &= |(ad-bc)z - (ad-bc)w| \\ &= |z-w|. \end{aligned}$$

Together with equation (1.2) we get

$$\begin{aligned} \frac{|gz-gw|^2}{\mathrm{Im}(gz)\mathrm{Im}(gw)} &= \frac{|gz-gw|^2}{\frac{\mathrm{Im}(z)}{|cz+d|^2}\frac{\mathrm{Im}(w)}{|cw+d|^2}} \\ &= \frac{|cz+d|^2 \cdot |cw+d|^2 \cdot |gz-gw|^2}{\mathrm{Im}(z)\mathrm{Im}(w)} \\ &= \frac{|z-w|^2}{\mathrm{Im}(z)\mathrm{Im}(w)}. \end{aligned}$$

Hence we conclude that the right hand side of (1.4) is indeed $\mathrm{SL}_2(\mathbb{R})$ invariant.

Given two points $z, w \in \mathbb{H}$, there is a unique orthogonal circle containing z and w. By Lemma 1.6 we can map this orthogonal circle with some element $g \in \mathrm{SL}_2(\mathbb{R})$ to the imaginary axis. This implies by g invariance of (1.4) that it suffices to check the equation for points $z = y_0 i$ and $w = y_1 i$ with $0 < y_0 < y_1$. By Lemma 1.8 we conclude

$$\begin{aligned} \cosh(d(z,w)) &= \cosh(\log(y_1) - \log(y_2)) \\ &= \frac{1}{2}\left(\frac{y_1}{y_2} + \frac{y_2}{y_1}\right) \\ &= 1 + \frac{(y_2-y_1)^2}{2y_1y_2} = 1 + \frac{|z-w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)}. \end{aligned}$$

□

Theorem 1.10. *The oriented isometry group of $\mathbb{H}$ is*

$$\mathrm{Iso}^+(\mathbb{H}) \cong \mathrm{PSL}_2(\mathbb{R}).$$

Proof. The group homomorphism is given by associating to each element of $\mathrm{PSL}_2(\mathbb{R})$ the isometry defined by equation (1.1). It is obvious that this group homomorphism is injective. For surjectivity we refer to [Bea91] chapter 7.4. □

Definition 1.11. A *global geodesic* is the image of a smooth curve $\phi : \mathbb{R} \to \mathbb{H}$ with the property that for all $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$,

$$L(\phi|_{[t_1,t_2]}) = d(\phi(t_1), \phi(t_2)).$$

Theorem 1.12. *The global geodesics are precisely the orthogonal circles.*

Proof. By Lemma 1.8 it follows that the imaginary axis in $\mathbb{H}$ is a global geodesic. By Lemma 1.1 every orthogonal circle can be written as $g \circ A$ for A the imaginary axis in $\mathbb{H}$. Since the action is isometric, this implies that every orthogonal circle is a global geodesic. Conversely, we can map every orthogonal circle to the imaginary axis by an element of $\mathrm{SL}_2(\mathbb{R})$. So again the action being an isometric group action implies that every orthogonal circle is a geodesic. □

Euclid's fifth axiom states that for every line ℓ and every point p outside of ℓ, there is a unique line passing through p that is parallel to ℓ. More precisely, we define a line to be parallel to another line ℓ if the line does not intersect ℓ. For a long time, many mathematicians tried to derive the fifth axiom from the other four. However they failed. As Figure 6 shows, the hyperbolic plane, together with global geodesics as lines, does not satisfy this famous axiom since there are many parallel lines through a point p outside a given line. Hence the fifth axiom is independent of the others and there is no proof that deduces the fifth axiom from the remaining four.

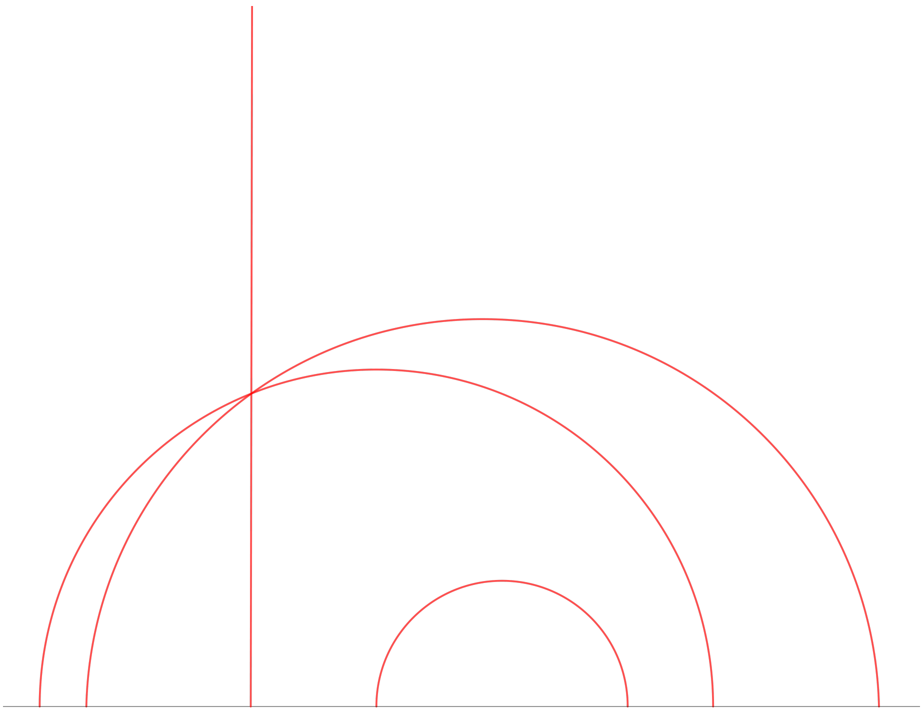

Figure 6: Hyperbolic space does not satisfy Euclid's fifth axiom

We next classify the orientation preserving isometries according to the number and the location of their fixed points. Denote

$$\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \qquad \text{and} \qquad \widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}.$$

For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ we define

$$g \circ \infty = \frac{a}{c}$$

and furthermore define $\frac{a}{c} := \infty$ whenever $c = 0$. With this convention ∞ is a fixed point for g whenever $c = 0$. If $z \in \mathbb{C}$, then $gz = z$ is equivalent to

$$cz^2 + (d - a)z - b = 0. \tag{1.5}$$

If $c = 0$, then the equation has at most one zero in $\mathbb{C}$. For the other case that $c \neq 0$, the equation has at least one but no more than two distinct zeros in the complex numbers. To summarize, any element $g \in \mathrm{PSL}_2(\mathbb{R})$ not equal to the identity has either one or two fixed points in $\widehat{\mathbb{C}}$. Additionally, note that if z is a zero of the above equation, then so is $\overline{z}$. This allows the following classification of elememts of $\mathrm{PSL}_2(\mathbb{R})$ not equal to the identity.

Definition 1.13. An element $g \in \mathrm{PSL}_2(\mathbb{R})$ is called

(i) *parabolic* if g has exactly one fixed point in $\hat{\mathbb{R}}$. Hence the fixed point is contained in $\widehat{\mathbb{R}}$.

(ii) *hyperbolic* if g has two distinct fixed points in $\widehat{\mathbb{R}}$.

(iii) *elliptic* if g has a fixed point $z \in \mathbb{H}$ and hence $\overline{z}$ is a second distinct fixed point.

Proposition 1.14. *An element $g \in \mathrm{PSL}_2(\mathbb{R})$ not equal to the identity is*

(i) parabolic whenever g is conjugate to $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ for $b \neq 0$, and this holds if and only if $|\mathrm{tr}(g)| = 2$.

(ii) hyperbolic whenever g is conjugate to $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ with $\lambda > 1$, and this holds if and only if $|\mathrm{tr}(g)| > 2$.

(iii) elliptic whenever g is conjugate to an element of $\mathrm{PSO}_2(\mathbb{R})$, and this holds if and only if $|\mathrm{tr}(g)| < 2$.

Proof. We only prove (i) and omit the cases (ii) and (iii) since they follow from similar arguments. Assume that $g \in \mathrm{PSL}_2(\mathbb{R})$ is a parabolic element with fixed point $p \in \widehat{\mathbb{R}}$. Denote by g' an element of $\mathrm{PSL}_2(\mathbb{R})$ such that $gp = \infty$. Then the element

$$g^* = g' g g'^{-1}$$

has ∞ as the only fixed point. We write $g^* = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. By equation (1.5), we conclude $c = 0$ and $a = d$. Thus $a = d = 1$ and hence g^* is of the form $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ for some b. We have $b \neq 0$ since g is not the identity and the conjugacy class of the identity element consists of the identity element itself. Hence g is conjugate to $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ for some $b \neq 0$. This condition also implies that the absolute value of the trace of g is equal to two, since conjugation leaves the trace invariant.

Assume now for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{R})$ that $|\mathrm{tr}(g)| = 2$. We can assume $a + d = 2$ and that $a \neq 0$. Furthermore up to a conjugation by a matrix of the form $\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)$ we can assume that $c = 0$. Hence, as $g \in \mathrm{SL}_2(\mathbb{R})$, $ad = a(2-a) = 1$. Thus $a = 1$ and so $d = 1$. Furthermore, because g is by assumption not the identity, we conclude that g is conjugate to an element of the form $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ for $b \neq 0$. □

1.2 Geodesics on Hyperbolic Surfaces

In this section we study hyperbolic surfaces and geodesics on these spaces. We first simply consider a discrete subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$. We define

$$S := \Gamma\backslash\mathbb{H} := \{\Gamma z \,:\, z \in \mathbb{H}\}.$$

We call a space of the form $S = \Gamma\backslash\mathbb{H}$ a *quotient of hyperbolic space*. Discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ are called *Fuchsian groups*.

The first lemma in this section gives us a natural measure on $\mathbb{H}$. We note that this measure is the projection of the Haar measure on $\mathrm{PSL}_2(\mathbb{R})$ via the projection map $\mathrm{PSL}_2(\mathbb{R}) \to \mathbb{H} = \mathrm{PSL}_2(\mathbb{R})/\mathrm{PSO}_2(\mathbb{R})$.

Lemma 1.15. *The measure*

$$\mu_{\mathbb{H}}(z) := \frac{1}{y^2} dx dy,$$

where $z = x + iy$, is a $\mathrm{PSL}_2(\mathbb{R})$-invariant measure on $\mathbb{H}$.

Proof. We want to show that for any continuous function $f : \mathbb{H} \to \mathbb{R}$ and any $g \in \mathrm{PSL}_2(\mathbb{R})$ we have

$$\int_{\mathbb{H}} f(gz)\, d\mu_{\mathbb{H}}(z) = \int_{\mathbb{H}} f(z)\, d\mu_{\mathbb{H}}(z).$$

The transformation $z \mapsto gz$ has derivative $\frac{1}{(cz+d)^2}$ and hence the Jacobian of the transformation is $\frac{1}{|cz+d|^4}$. Thus, by using the transformation formula,

$$\begin{aligned}\int_{\mathbb{H}} f(gz)\, \frac{1}{y^2} dx dy &= \int_{\mathbb{H}} f(gz)\, \frac{\frac{1}{|cz+d|^4}}{\frac{y^2}{|cz+d|^4}} dx dy \\ &= \int_{\mathbb{H}} f(z') \frac{1}{y(z')^2} dx' dy'.\end{aligned}$$

□

We next discuss an equivalent characterization for Γ being a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$.

Definition 1.16. Let Γ be a group acting by isometries on $\mathbb{H}$. The action of Γ is said to be *properly discontinuous* if for any compact set $K \subset \mathbb{H}$, the set

$$\{\gamma \in \Gamma \,:\, \gamma K \cap K \neq \emptyset\}$$

is finite.

Proposition 1.17. *A subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$ is discrete if and only if its action on $\mathbb{H}$ is properly discontinuous.*

Proof. If Γ is not discrete, there exists a sequence of elements $\gamma_n \in \Gamma$ with $\gamma_n \neq e$ such that $\gamma_n \to e$ as $n \to \infty$. If K is a compact subset of $\mathbb{H}$ containing an open set, it follows that for sufficiently large n the set $\gamma_n K \cap K$ is not empty. Hence the action is not properly discontinuous.

Assume next that Γ is discrete and let $K \subset \mathbb{H}$ be a compact subset. If we show that $B = \{g \in \mathrm{SL}_2(\mathbb{R}) : gK \cap K \neq \emptyset\}$ is compact, then by discreteness of Γ we conclude that the set $\{\gamma \in \Gamma : \gamma K \cap K \neq \emptyset\}$ is finite.

The set B is closed, hence it suffices to prove that B is a bounded set in $\mathrm{SL}_2(\mathbb{R})$, where we view $\mathrm{SL}_2(\mathbb{R})$ as a subset of $\mathbb{R}^4$. By compactness of K, there are constants $R, \varepsilon > 0$ such that every $w \in K$ has $|w| \leq R$ and $\mathrm{Im}(w) \geq \varepsilon$. It follows that if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in B$, then $gz \in K$ for some $z \in K$. Hence $|\frac{az+b}{cz+d}| \leq R$ and $\mathrm{Im}(\frac{az+b}{cz+d}) = \frac{\mathrm{Im}(z)}{|cz+d|^2} \geq \varepsilon$. Thus

$$|cz+d|^2 \leq \frac{1}{\varepsilon}\mathrm{Im}(z) \leq \frac{R}{\varepsilon}$$

and

$$|az+b|^2 \leq R^2 |cz+d|^2 \leq \frac{R^3}{\varepsilon}.$$

Since z is contained in some compact set of $\mathbb{H}$, this implies that the coefficients of the matrices in B lie in a bounded subset of $\mathbb{R}^4$. $\square$

We next introduce the notion of a *fundamental domain.* For a given quotient of hyperbolic space $S = \Gamma\backslash\mathbb{H}$, these are subsets of $\mathbb{H}$ that represent S in $\mathbb{H}$.

Definition 1.18. Let Γ be a Fuchsian group. A measurable set $F \subset \mathbb{H}$ is called a *fundamental domain* for Γ if the following two properties hold:

(i) If $\gamma_1, \gamma_2 \in \Gamma$ are not equal, then $\gamma_1 F \cap \gamma_2 F = \emptyset$.

(ii) $\bigcup_{\gamma \in \Gamma} \gamma \overline{F} = \mathbb{H}$, where $\overline{F}$ denotes the closure of F.

Lemma 1.19. *The measure of any two fundamental domains for Γ is equal.*

Proof. Let F_1 and F_2 be fundamental domains for Γ. Then by invariance of the measure $\mu_{\mathbb{H}}(z)$,

$$\mu_{\mathbb{H}}(F_1) = \mu_{\mathbb{H}}\left(F_1 \cap \bigcup_{\gamma\in\Gamma} \gamma F_2\right) = \mu_{\mathbb{H}}\left(\bigcup_{\gamma\in\Gamma} \gamma^{-1} F_1 \cap F_2\right) = \mu_{\mathbb{H}}(F_2).$$

$\square$

A particular nice class of fundamental domains are given by so-called Dirichlet domains.

Definition 1.20. Let Γ be a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ and let $p \in \mathbb{H}$ be a point not fixed by any element of Γ other than the identity. Then the set

$$D_p := \{z \in \mathbb{H} : d(z,p) < d(\gamma z, p) \text{ for all } \gamma \in \Gamma\backslash\{e\}\}$$

is called *Dirichlet domain.*

We show in the next proposition that Dirichlet domains are indeed fundamental domains. As a matter of fact, Dirichlet domains are geodesic polygons, i.e. a region where the boundary consists of geodesic segments. For a proof of this fact we refer to Lemma 11.5 of [EW11]. Thus, we can roughly think of a quotient of hyperbolic space as a geodesic polygon in $\mathbb{H}$.

Proposition 1.21. *Any Dirichlet domain for Γ is a fundamental domain.*

Proof. A Dirichlet region is open and hence measurable. We check the two properties. First assume that $\gamma_1, \gamma_2 \in \Gamma$ are not equal. We want to show $\gamma_1 D_p \cap \gamma_2 D_p = \emptyset$. Assume that this is not the case. Then there are elements $z_1, z_2 \in D_p$ such that $\gamma_1 z_1 = \gamma_2 z_2$, or equivalently $\gamma_2^{-1}\gamma_1 z_1 = z_2$. By definition of D_p, for $\gamma = \gamma_2^{-1}\gamma_1$,

$$d(z_1, p) < d(\gamma z_1, p) = d(z_2, p) < d(\gamma^{-1} z_2, p) < d(z_1, p),$$

a contradiction.

For the second condition we want to show that for any $z \in \mathbb{H}$, there is some $\gamma \in \Gamma$ such that $\gamma z \in \overline{D_p}$. We assume without loss of generality that $z \notin D_p$. Since the action is properly discontinuous there are only finitely many γ such that

$$d(\gamma z, p) \leq d(z, p) + 1.$$

Hence there is a γ such that $d(\gamma z, p) \leq d(\gamma' z, p)$ for all $\gamma' \in \Gamma$. Thus $\gamma z \in \overline{D}_p$. $\square$

Definition 1.22. A *lattice* is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ such that the measure of any fundamental domain for $\Gamma\backslash\mathbb{H}$ is finite. A lattice is called *cocompact* if $\Gamma\backslash\mathbb{H}$ is compact, or equivalently that there is a fundamental domain with compact closure.

Up to now, we only viewed a quotient of hyperbolic space as a topological space endowed with the quotient topology. A natural question to ask is under which conditions for Γ, the space $\Gamma\backslash\mathbb{H}$ forms a smooth surface. This question is partially answered by the following proposition from [Bur16].

Proposition 1.23. *Let Γ be a group acting freely and properly discontinuously on a manifold M by diffeomorphisms. Then there is a unique smooth manifold structure on $\Gamma\backslash M$ such that $p : M \to \Gamma\backslash M$ is a smooth covering.*

Proof. [Bur16] pages 22 and 23. $\square$

In our concrete case we conclude that $\Gamma\backslash\mathbb{H}$ is a smooth surface if Γ does not contain any elliptic elements, since these are the only ones which have fixed points in $\mathbb{H}$. This motivates the following definition.

Definition 1.24. Let Γ be a Fuchsian group. Then the quotient of hyperbolic space $S = \Gamma\backslash\mathbb{H}$ is called a *hyperbolic surface* if Γ does not contain any elliptic elements.

Remark. From the viewpoint of Riemannian geometry one defines a hyperbolic surface as a smooth complete surface of constant curvature -1. Since $\mathbb{H}$ is up to isometry the unique simply connected surface of constant curvature -1, it follows that every hyperbolic surface is isometric to $\Gamma\backslash\mathbb{H}$ with Γ a Fuchsian group with no elliptic elements. Hence the above definition covers all hyperbolic surfaces.

We can further relate the property of $S = \Gamma\backslash\mathbb{H}$ being compact to the number and location of the fixed points of the elements of Γ. For this recall the classification of isometries of $\mathbb{H}$ carried out at the end of section 1.1.

Proposition 1.25. *If $\Gamma\backslash\mathbb{H}$ is a compact hyperbolic surface, then Γ consists only of hyperbolic elements and the identity.*

Proof. We first claim that if $\Gamma\backslash\mathbb{H}$ is compact, then for all $\gamma \in \Gamma$ the conjugacy class

$$C_G(\gamma) = \{g\gamma g^{-1} \,:\, g \in\}$$

is closed, for $G = \mathrm{PSL}_2(\mathbb{R})$.

To prove this consider a converging sequence $g_n\gamma g_n^{-1} \to h$ for $g_n, h \in G$ in $C_G(\gamma)$. We want to show that $h \in C_G(\gamma)$. Since $\Gamma\backslash\mathbb{H}$ is compact, there is a fundamental domain $C \subset \mathbb{H}$ with compact closure. Write $g_n = c_n\gamma_n$ for $c_n \in \overline{C}$ and $\gamma_n \in \Gamma$. By compactness of c we may assume $c_n \to c \in \overline{C}$. Let V be a compact neighborhood of h such that $g_n\gamma g_n^{-1} = c_n\gamma_n\gamma\gamma_n^{-1}c_n^{-1} \in V$ for n large enough. Hence $\gamma_n\gamma\gamma_n^{-1} \in C^{-1}VC \cap \Gamma$. Note that $C^{-1}VC$ is compact and thus the set $C^{-1}VC \cap \Gamma$ is finite. Consequently, we can assume without loss of generality that $\gamma_n\gamma\gamma_n^{-1} = \gamma^* \in \Gamma$ for all n. This implies

$$h = \lim_{n\to\infty} c_n\gamma^* c_n^{-1} = c\gamma^* c^{-1} \in C_G(\gamma).$$

So we proved the claim.

Next observe that if $\gamma \in \Gamma\backslash\{I_2\}$ is parabolic, it is by Proposition 1.14 conjugate to some $\gamma_b = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ for some $b \neq 0$. Note that for $\lambda \neq 0$,

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & \lambda^2 b \\ 0 & 1 \end{pmatrix}.$$

Thus, if we choose $\lambda \to 0$, then the above conjugated element converges to the identity matrix. However, the identity matrix is not contained in the conjugacy class of any element that is not equal to the identity itself. Hence the conjugacy class of a parabolic element is not closed and thus by the first claim, Γ does not contain any parabolic elements. Further, by definition, Γ does not contain any elliptic elements. Hence Γ consists of hyperbolic elements and the identity. $\square$

Thus, we finally arrived at the central object of study in this thesis. Namely, compact hyperbolic surfaces $S = \Gamma\backslash\mathbb{H}$. These are by the last proposition simply described by hyperbolic surfaces $S = \Gamma\backslash\mathbb{H}$, where Γ only consists of hyperbolic elements and the identity. We next discuss geodesics and in particular, closed geodesics on hyperbolic surfaces.

Definition 1.26. Let $S = \Gamma\backslash\mathbb{H}$ be a hyperbolic surface and $\pi : \mathbb{H} \to S$ the projection. A *geodesic on S* is the image of a global geodesic on $\mathbb{H}$ under π.

In order to define the notion of *closed geodesics*, recall that we call a continuous curve $\phi : \mathbb{R} \to \Gamma\backslash\mathbb{H}$ *closed* if there exists some $T \in \mathbb{R}$ such that

$$\phi(t) = \phi(t + T)$$

for all $t \in \mathbb{R}$. We call a curve $\phi : \mathbb{R} \to \Gamma\backslash\mathbb{H}$ *trivial* if for all $t \in \mathbb{R}$ we have $\phi(t) = p$ for some fixed $p \in \mathbb{H}$.

Definition 1.27. A *closed geodesic on a hyperbolic surface* $\Gamma\backslash\mathbb{H}$ is a tuple (γ, T) where $\gamma : \mathbb{R} \to \Gamma\backslash\mathbb{H}$ is a continuous smooth curve that is a geodesic on $\Gamma\backslash\mathbb{H}$ and a non-trivial closed curve. Further, $T > 0$ is a real number such that $\gamma(t) = \gamma(t+T)$ for all $t \in \mathbb{R}$.

We define the *length* of the closed geodesic (γ, T) as follows. If $\tilde{\gamma} : \mathbb{R} \to \mathbb{H}$ is a geodesic on $\mathbb{H}$ such that $\pi(\tilde{\gamma}(t)) = \gamma(t)$, then the length of γ is the distance between $\tilde{\gamma}(t)$ and $\tilde{\gamma}(t+T)$.

Remark. Note that the length of a closed geodesic is well defined since any two lifts of a closed geodesic are the same up to an isometry. Furthermore, it is important to remark that with this definition we distinguish a closed geodesic and iterates of the same closed geodesic. Closed geodesics that are not iterates of another closed geodesic are called *prime* or *primitive*.

We say two geodesics (γ, T) and (γ', T') are *equal* if there is a reparametrization $r : \mathbb{R} \to \mathbb{R}$ such that $\gamma(r(t+T)) = \gamma'(t+T')$ for all $t \in \mathbb{R}$. In this sense, every statement concerning the uniqueness of a certain closed geodesic or regarding the number of a certain set of closed geodesics is to be understood up to reparametrization.

Lemma 1.28. *Let g be a hyperbolic element of* $\mathrm{PSL}_2(\mathbb{R})$*. Then there exists a unique global geodesic on $\mathbb{H}$ that is preserved by g.*

Proof. By Proposition 1.14, g is conjugate to $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ for some $\lambda > 1$. So there is some g' such that

$$g' g g'^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

The matrix $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ fixes the points 0 and ∞ and preserves the unique global geodesic ϕ connecting 0 and ∞. We therefore have that the geodesic $g'^{-1}\phi$ is preserved by g. The uniqueness follows from the fact, that a hyperbolic element preserves two elements of $\widehat{\mathbb{R}}$ and that there is a unique geodesic connecting these two points. $\square$

In the setting of the above proof, let z be an element of the unique geodesic that is preserved by g. Then $g'^{-1}z$ lies on the imaginary axis. Hence

$$\begin{aligned} d(z, gz) &= d(g'z, g'gz) \\ &= d(g'z, g'^{-1}\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)z) \\ &= d(g'z, \lambda^2 g'z) \\ &= \log\bigl(\lambda^2\bigr) \\ &= 2\log(\lambda). \end{aligned}$$

Definition 1.29. For a hyperbolic element g of $\mathrm{PSL}_2(\mathbb{R})$, the unique global geodesic on $\mathbb{H}$ that is preserved by g is called the *axis* of g. We denote the axis of g by a_g.

If g is conjugate to $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ for $\lambda > 0$, then we call the real number

$$\ell(g) = 2\log(\lambda)$$

the *displacement length* of g.

Lemma 1.30. *Let $\Gamma\backslash\mathbb{H}$ be a compact hyperbolic surface. Then for any element $g \in \Gamma$ the image of the axis of g under the projection $\mathbb{H} \to \Gamma\backslash\mathbb{H}$ is a closed geodesic on $\Gamma\backslash\mathbb{H}$ of length $\ell(g)$.*

Proof. Let $g \in \Gamma$ and let a_g be the axis of g. Denote by $\gamma : \mathbb{R} \to \mathbb{H}$ a parametrization of a_g such that

$$d(\gamma(t + t_0), \gamma(t_0)) = t$$

for all $t_0, t \in \mathbb{R}$. Hence

$$\gamma(t + \ell(g)) = g\gamma(t)$$

in $\mathbb{H}$. Denote by $\pi : \mathbb{H} \to \Gamma\backslash\mathbb{H}$ the projection. Consequently, $\pi(\gamma(t)) : \mathbb{R} \to \Gamma\backslash\mathbb{H}$ is a parametrization of the geodesic in $\Gamma\backslash\mathbb{H}$. Thus

$$\pi(\gamma(t + \ell(g))) = \pi(\gamma(t)),$$

implying that the geodesic is closed. The length of the closed geodesic is furthermore equal to $\ell(g)$ since the axis is the unique global geodesic that is fixed by g. $\square$

We are now ready to prove the central theorems in this section.

Theorem 1.31. *Let $S = \Gamma\backslash\mathbb{H}$ be a compact hyperbolic surface. Every loop on S is freely homotopic to a unique closed geodesic.*

Proof. Let ϕ be a non-trivial closed curve and denote by $\tilde{\phi}$ a maximal continuous curve in $\mathbb{H}$ obtained by joining successive lifts of ϕ. Hence there is some $g \in \Gamma$ such that g preserves $\tilde{\phi}$. By assumption, g is hyperbolic. The axis of g descends to a closed geodesic γ on $\Gamma\backslash\mathbb{H}$ in the free homotopy class of ϕ. Since a_g is the unique geodesic fixed by g there is no other geodesic in the free homotopy class of γ. $\square$

Theorem 1.32. *Let $S = \Gamma\backslash\mathbb{H}$ be a compact hyperbolic surface. Then the closed geodesics of S correspond precisely to conjugacy classes of elements of Γ.*

Proof. Let (γ, T) be a closed geodesic and let $\widetilde{\gamma} : \mathbb{R} \to \mathbb{H}$ be a global geodesic on $\mathbb{H}$ that is a lift of γ. Assume without loss of generality that the parametrization of $\widetilde{\gamma}$ is of unit speed, meaning that for all $t, t_0 \in \mathbb{R}$ we have $d(\widetilde{\gamma}(t + t_0), \widetilde{\gamma}(t_0)) = t$. There is an element $g \in \Gamma$ that preserves $\widetilde{\gamma}$. By Lemma 1.28 $\widetilde{\gamma}$ is equal to the axis a_g of g. Furthermore, from Lemma 1.30 it follows that the length of the geodesic γ is equal to the displacement length $\ell(g)$. Thus, from the proof of Lemma 1.28 is follows that for any $g' \in \mathrm{PSL}_2(\mathbb{R})$,

$$a_{g'gg'^{-1}} = g'^{-1} a_g. \tag{1.6}$$

Since image of $g'^{-1}a_g$ under the projection $\mathbb{H} \to \Gamma\backslash\mathbb{H}$ is the same as the one of a_g, it follows that the geodesic corresponds to the conjugacy class of g. The converse correspondence follows analogously from equation (1.6). $\square$

Theorem 1.33. *Let $S = \Gamma\backslash\mathbb{H}$ be a compact hyperbolic surface. Fix a real number $L > 0$. There are at most finitely many closed geodesics on S of length smaller or equal than L.*

Proof. Let γ be a closed geodesic on $\Gamma\backslash\mathbb{H}$ of length less than L. Let $g \in \Gamma$ be the unique element that preserves a global geodesic $\widetilde{\gamma}$ on $\mathbb{H}$ that lifts γ. By Lemma 1.30 the length of γ is equal to the displacement length $\ell(g) = d(x, gx)$ for all $x \in \widetilde{\gamma}$. Fix some $x \in \widetilde{\gamma}$.

Let K be the compact closure of a fundamental domain for Γ. Hence there exists an element $g' \in \Gamma$ such that $y := g'x \in K$. Denote $g^* := g'gg'^{-1}$. Then

$$d(y, g^*y) = d(g'x, g'gg'^{-1}y) = d(x, gx) = \ell(g).$$

Write

$$d(K, g^*K) := \inf_{k_1,k_2\in K} d(k_1, g^*k_2) \qquad \text{and} \qquad d_K := \sup_{k_1,k_2\in K} d(k_1, k_2).$$

Hence, if $d(K, g^*K) \leq 2d_K + L$ for some $g \in \Gamma$, then the geodesic corresponding to the conjugacy class of g^* has length less than L. Thus the statement of the theorem follows if we show that the set

$$F := \{g \in \Gamma \,:\, d(K, gK) \leq L'\}$$

for $L' = d_K + L$ is finite.

We will show that this follows from the assumption that Γ acts properly discontinuously. Denote by K' the compact set defined by

$$K' = \{k' \in \mathbb{H} \,:\, d(k', k) \leq \tfrac{L'}{2} \text{ for some } k \in K\}.$$

If there is some $g \in \Gamma$ such that $K' \cap gK' \neq \emptyset$, then there are some $k_1', k_2' \in K'$ such that $k_1' = gk_2'$. By definition, we have $k_1, k_2 \in K$ such that $d(k_i', k_i) \leq \frac{L'}{2}$ for $i = 1, 2$. Hence

$$\begin{aligned} d(K, gK) &\leq d(k_1, gk_2) \\ &\leq d(k_1, k_1') + d(k_1', gk_2') + d(gk_2', gk_2) \\ &\leq L'. \end{aligned}$$

Thus if $K' \cap gK' \neq \emptyset$, then $g \in F$. Since the action is properly discontinuous, we conclude that the set of elements such that $K' \cap gK' \neq \emptyset$ is finite and hence the set F is finite. ☐

For a set M, we denote by $\#M$ the cardinality of M. So we proved that for all $L > 0$, the number

$$c_L(S) := \#\{\text{closed geodesics on } S \text{ with length } \leq L\}$$

is finite. It is thus sensible to ask how $c_L(S)$ behaves as L tends to infinity. We will show in Corollary 3.16 that

$$c_S(L) \sim \frac{e^L}{L}$$

as L tends to infinity, where we write $f(L) \sim g(L)$ for two real valued functions f and g whenever $\frac{f(L)}{g(L)} \to 1$ as L tends to infinity. We can already prove that $c_S(L)/e^L$ is bounded as the next lemma shows.

Lemma 1.34. *For S a compact hyperbolic surface,*

$$c_S(L) = O(e^L)$$

as $L \to \infty$.

Proof. Let $x \in S$ and denote by w a lift of x in $\mathbb{H}$ and by D_w the Dirichlet fundamental domain for w. If γ is a closed geodesic on S of length less than L, denote by $\tilde{\gamma}$ a lift of the geodesic that passes through some point $q \in D_w$. $\tilde{\gamma}$ is the axis of some element $g \in \Gamma$. Denote by a the diameter of the Dirichlet fundamental domain D_w. Then

$$d(w, gw) \leq d(w, q) + d(q, gq) + d(gq, gw) \leq L + 2a.$$

Hence for each closed geodesic of length less L there is an element $g \in \Gamma$ that maps w to a point of distance less than $L + 2a$ away. Hence $c_S(L)$ is bounded by the number of images of D_w lying within a distance of $L + 3a$ to the point w, so we get the bound

$$c_S(L) \leq \frac{\mu_{\mathbb{H}}(B(w, L + 3a))}{\mu_{\mathbb{H}}(Z)}.$$

Lastly note that

$$\begin{aligned} \mu_{\mathbb{H}}(B(w, L + 3a)) &= \int_{B(w,L+3a)} 1 \, d\mu_{\mathbb{H}}(z) \\ &= 2\pi \int_0^{L+3a} \sinh(r) \, dr \\ &= 2\pi(\cosh(L + 3a) - 1) \\ &= \pi(e^{L+3a} + e^{-(L+3a)} - 1) \\ &\leq (2\pi e^{3a})e^L = O(e^L), \end{aligned}$$

where we used geodesic polar coordinate in the second lines (for more details see appendix A of [Ber11]). □

We can further discern the elements of Γ that correspond to primitive closed geodesics. This will later turn out to be useful since the number of primitive closed geodesics behaves asymptotically as the number of closed geodesics. We shall call an element $\delta \in \Gamma\backslash\{I\}$ *primitive* if it cannot be written as a non-trivial power of an element in Γ. The corresponding closed geodesic is then a primitive closed geodesic.

Proposition 1.35. *Let Γ be a Fuchsian group consisting of hyperbolic elements. For all $g \in \Gamma\backslash\{I\}$, there is a unique primitive element $\delta \in \Gamma$ such that $g = \delta^n$ for a certain $n \geq 1$. Furthermore, the centralizer of g in Γ is*

$$\Gamma_g = \{\delta^n \, : \, n \in \mathbb{Z}\}.$$

Proof. Let a_g be the axis of Γ. We denote by Z the subgroup of Γ which fixes the axis a_g. Then Z acts freely and properly discontinuously on a_g. The restriction $Z|_{a_g}$ can therefore be viewed as a discrete subgroup of $\mathbb{R}$ by associating to each element of $Z|_{a_g}$ its displacement length. There is thus $\delta \in Z$ such that $\ell(\delta) > 0$ and $\ell(\delta) \leq \ell(g)$ for all $g \in Z \backslash \{I\}$. Hence there is a unique $n \in \mathbb{Z}$ such that $n\ell(\delta) = \ell(g)$ and so $\gamma = \delta^n$. By replacing δ with δ^{-1}, we can assume $m > 0$. The fact that $Z = \Gamma_g$ follows from the proof of Theorem 1.31. □

2 The Spectrum of the Laplacian on Compact Hyperbolic Surfaces

The aim of this section is to study the Laplace operator on compact hyperbolic surfaces and to prove that there is a countable number of eigenvalues. There are several approaches for investigating the Laplace operator. In [Bus92] and [Ber11] a direct approach based on the construction of heat kernels is performed. We however choose an method based on representation theory. In the following paragraphs, we describe the strategy we will use in this section in order to understand the Laplace operator.

In section 2.1 we consider for a general unimodular topological group G and Γ a cocompact subgroup, the space $L^2_\mu(\Gamma\backslash G)$ together with the representation given by right multiplication, which is called the regular right representation. The main result of section 2.1 is a decomposition of the space $L^2_\mu(\Gamma\backslash G)$ into a direct sum of a countable number of irreducible subrepresentations. As a consequence, for K a compact subgroup of G, we can also decompose the space $L^2_\mu(\Gamma\backslash(G/K))$.

In the following section, we study so called Gelfand pairs (G, K). We will see that every Riemannian symmetric pair forms a Gelfand pair. We prove for Gelfand pairs (G, K) that the irreducible subspaces in the above mentioned decomposition of $L^2_\mu(\Gamma\backslash(G/K))$ are one-dimensional. Hence, for each of these subspaces we can choose a generating element. In the case of $(G, K) = (\mathrm{PSL}_2(\mathbb{R}), \mathrm{PSO}_2(\mathbb{R}))$ we want to prove that these generators are precisely the eigenfunctions of the Laplace operator and hence we get a decomposition of $L_\mu(\Gamma\backslash\mathbb{H})$ into eigenfunctions of the Laplace operator.

In order to carry this out, in section 2.3 we relate the above mentioned functions that generate irreducible subspaces of $L^2_\mu(\Gamma\backslash(G/K))$ to a special class of bi-K-invariant functions. More precisely, we are interested in so-called spherical functions that are closely related to characters the Banach algebra of bi-K-invariant $L^1_\mu(G)$-functions. We can then relate the functions from the decomposition of $L^2_\mu(\Gamma\backslash(G/K))$ to eigenfunctions of operators related to spherical functions.

In section 2.4 we finally consider the particular case of our interest, namely $(G, K) = (\mathrm{PSL}_2(\mathbb{R}), \mathrm{PSO}_2(\mathbb{R}))$. In this setting we have that bi-K-invariant functions on G are directly related to functions on $\mathbb{H}$ that only depend on the distance to $i \in \mathbb{H}$. We can equivalently study so-called point pair invariant functions $k : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ that only depend on the distance of their input variables. Since spherical functions are also bi-K-invariant, the point pair invariant functions also relate to spherical functions. Furthermore, point pair invariant function have useful properties concerning the Laplace operator. So we use these properties together with the results from section 2.3 to prove that there is an orthonormal basis of eigenfunctions of the Laplace operator for compact hyperbolic surfaces. We conclude this section by stating some properties of the eigenvalues without proof.

The main reference for this section is [Bur07]. Most of the content of this section can be found in one of the three books [GGPS69], [Bum98] and [Ber11].

2.1 Decomposition of $L^2_\mu(\Gamma\backslash G)$

In this section we mostly follow [GGPS69].

Denote by G a unimodular locally compact group with Haar measure μ. We thus have for all Borel measurable integrable functions $f : G \to \mathbb{C}$ and $h \in G$,

$$\int_G f(g)\,d\mu(g) = \int_G f(hg)\,d\mu(g) = \int_G f(gh)\,d\mu(g).$$

Recall that the space $L^2_\mu(G)$ forms a Hilbert space with inner product defined for $f_1, f_2 \in L^2_\mu(G)$ by

$$\langle f_1, f_2\rangle = \int_G f_1(g)\overline{f_2(g)}\,d\mu(g).$$

The scalar product induces a norm on $L^2_\mu(G)$ given by

$$||f||_2 = \sqrt{\langle f, f\rangle}.$$

We furthermore consider the following unitary representation $\pi : G \to U(L^2_\mu(G))$. For each $g \in G$ the operator $\pi(g)$ is given by

$$\pi(g) : L^2_\mu(G) \to L^2_\mu(G), \qquad f \mapsto \pi(g)f = f \circ r_g,$$

where r_g is right multiplication by some element $g \in G$. Hence $(\pi(g)f)(x) = f(xg)$. We call this representation the *regular right representation*. The regular right representation is unitary, as we prove in the appendix (see appendix B for a definition of unitary representations and Proposition B.25 for a proof of this statement).

Consider next $\Gamma < G$, a discrete subgroup. We call Γ *cocompact* if $\Gamma\backslash G$ is compact. As G is unimodular and Γ is discrete, there is a unique Borel measure $\mu_{\Gamma\backslash G}$ such that for all compactly supported continuous functions $f \in C_c(G)$ we have that

$$\int_G f(g)\,d\mu(g) = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} g(\gamma g)\,d\mu_{\Gamma\backslash G}(G)$$

A proof of this can be found in [IZ17] in section 1.4. By a slight abuse of notation, we will drop the subscript $\Gamma\backslash G$ and will just write μ for this measure on $\Gamma\backslash G$. Note that the space $L^2_\mu(\Gamma\backslash G)$ is again a Hilbert space. Furthermore, a function $f : \Gamma\backslash G \to \mathbb{C}$ can be viewed as a Γ-invariant function $f : G \to \mathbb{C}$. Thus the regular right representation descends to a unitary representation on $L^2_\mu(\Gamma\backslash G)$.

Recall the following terminology. If $(\mathscr{H}, \pi)$ is a representation, we call a subspace $\mathscr{H}' \subset \mathscr{H}$ *invariant* if

$$\pi(g)\mathscr{H}' \subset \mathscr{H}',$$

for all $g \in G$. Hence π defines a representation on the invariant subspace $\mathscr{H}'$. We further call the representation $(\mathscr{H}, \pi)$ *irreducible* if the only invariant subspaces of $\mathscr{H}$ are the zero space $\{0\}$ and the space $\mathscr{H}$ itself. The central objective of this section is to prove the following theorem.

Theorem 2.1. *Let $\Gamma < G$ be a cocompact discrete subgroup. Then the Hilbert space $L^2_\mu(\Gamma\backslash G)$ together with the regular right representation splits into a countable direct sum of invariant and irreducible subspaces, each of which has finite multiplicity.*

We will generalize the current setting by just considering a unitary representation π of G on a Hilbert space $\mathscr{H}$. For $\varphi \in L^1_\mu(G)$, consider the following bilinear form:

$$B_\varphi(f_1, f_2) := \int_G \varphi(g)\langle \pi(g) f_1, f_2\rangle \, d\mu(g)$$

for $f_1, f_2 \in \mathscr{H}$. Note that

$$|B_\varphi(f_1, f_2)| \leq \int_G |\varphi(g)| \cdot |\langle \pi(g) f_1, f_2\rangle| \, d\mu(g) \leq ||\varphi||_1 \cdot ||f_1||_{\mathscr{H}} \cdot ||f_2||_{\mathscr{H}}$$

where we used the Cauchy-Schwarz inequality in the second line and the fact that the action is unitary. By the Lax-Milgram Lemma (Lemma B.18 in the appendix) we conclude that there exists an operator $\pi(\varphi) : \mathscr{H} \to \mathscr{H}$ with $||\pi(\varphi)||_{\text{op}} \leq ||\varphi||_1$ and $\langle \pi(\varphi) f_1, f_2\rangle = B_\varphi(f_1, f_2)$. Hence we can write $\pi(\varphi) f_1$ as

$$\pi(\varphi) f_1 = \int_G \varphi(g)\pi(g) f_1 \, d\mu(g).$$

We also use the notation

$$\pi(\varphi) = \int_G \varphi(g)\pi(g) \, d\mu(g).$$

For $\varphi_1, \varphi_2 \in L^1_\mu(G)$, recall that the convolution of φ_1 and φ_2 is defined as

$$(\varphi_1 * \varphi_2)(x) := \int_G \varphi_1(g)\varphi_2(g^{-1}x) \, d\mu(g) = \int_G \varphi_1(xg)\varphi_2(g^{-1}) \, d\mu(g).$$

Note that convolution together with the 1-norm gives $L^1_\mu(G)$ the structure of a Banach algebra.

Lemma 2.2. *The map*

$$L^1_\mu(G) \to B(\mathscr{H}), \qquad \varphi \mapsto \pi(\varphi)$$

is a Banach algebra homomorphism.

Proof. We need to check for $\varphi_1, \varphi_2 \in L^1_\mu(G)$ and $\lambda_1, \lambda_2 \in \mathbb{C}$,

$$\pi(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1\pi(\varphi_1) + \lambda_2\pi(\varphi_2)$$

and

$$\pi(\varphi_1 * \varphi_2) = \pi(\varphi_1)\pi(\varphi_2).$$

The first assertion is clear. We prove the second statement.

$$\begin{aligned}
\pi(\varphi_1 * \varphi_2) &= \int_G \varphi_1 * \varphi_2(g)\pi(g) \, d\mu(g) \\
&= \int_G \int_G \varphi_1(h)\varphi_2(h^{-1}g) \, d\mu(h) \, \pi(g) d\mu(g) \\
&= \int_G \int_G \varphi_2(h^{-1}g)\pi(g) \, d\mu(g) \, \varphi_1(h) d\mu(h) \\
&= \int_G \int_G \varphi_2(g)\pi(hg) \, d\mu(g) \, \varphi_1(h) d\mu(h) \\
&= \int_G \varphi_1(h)\pi(h) \, d\mu(h) \int_G \varphi_2(g)\pi(g) \, d\mu(g) \\
&= \pi(\varphi_1)\pi(\varphi_2).
\end{aligned}$$

□

For Γ a discrete subgroup, we can also consider the operator $\pi(\varphi) : L^2_\mu(\Gamma\backslash G) \to L^2_\mu(\Gamma\backslash G)$ with the analogous definition

$$(\pi(\varphi)f)(x) = \int_G \varphi(g)\pi(g)f(x)\,d\mu(g) = \int_G \varphi(g)f(xg)\,d\mu(g).$$

If Γ is cocompact, then these operators have the following interesting property.

Lemma 2.3. *Let Γ be a cocompact discrete subgroup. Then for a function $\varphi \in C_c(G)$ of compact support the operator*

$$\pi(\varphi) : L^2_\mu(\Gamma\backslash G) \to L^2_\mu(\Gamma\backslash G), \qquad \pi(\varphi) = \int_G \varphi(g)\pi(g)\,d\mu(g)$$

is compact.

Proof. Let $f \in \mathscr{H}$, which we view as a Γ-invariant function on G. Then

$$\begin{aligned}(\pi(\varphi)f)(x) &= \int_G \varphi(g)f(xg)\,d\mu(g)\\ &= \int_G \varphi(x^{-1}g)f(g)\,d\mu(g)\\ &= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} \varphi(x^{-1}\gamma g)f(\gamma g)\,d\mu(g)\\ &= \int_{\Gamma\backslash G} K(x,g)f(g)\,d\mu(g),\end{aligned}$$

where

$$K(x,g) := \sum_{\gamma\in\Gamma} \varphi(x^{-1}\gamma g).$$

The function $K(x,g)$ is called the *Selberg kernel* and will turn out to be of central importance in the derivation of the Selberg trace formula.

Since φ is of compact support, $\varphi(x^{-1}\gamma g)$ can only be nonzero if $x^{-1}\gamma g \in \operatorname{supp}(\varphi)$ or equivalently $\gamma \in x \cdot \operatorname{supp}(\varphi) \cdot g^{-1}$. Since $\Gamma\backslash G$ is compact, we can assume that x and g vary over a compact fundamental domain. Hence there are only finitely many $\gamma \in \Gamma$ for which $\varphi(x^{-1}\gamma g)$ is not zero, independent of x and g. Thus $k(x,g)$ is a finite sum of continuous functions and so continuous. Thus $\pi(\varphi)$ is a continuous integral operator and this implies that it is a compact operator. □

The theorem we want to prove (Theorem 2.1) hence follows from the following more general theorem, for $\mathscr{H} = L^2_\mu(\Gamma\backslash G)$ and π the regular right representation of G.

Theorem 2.4. *Let $(\mathscr{H}, \pi)$ be a unitary representation of a locally compact group G on a separable Hilbert space $\mathscr{H}$ such that the operators $\pi(\varphi)$ are compact for all $\varphi \in C_c(G)$ of compact support. Then $\mathscr{H}$ splits into a countable direct sum of subspaces that are invariant and irreducible, each of which has finite multiplicity.*

Proof. We first consider functions of compact support $\varphi \in C_c(G)$ such that

$$\varphi(g) = \overline{\varphi(g^{-1})}.$$

Then $\pi(\varphi)$ is self-adjoint since

$$\begin{aligned}\pi(\varphi)^* &= \int_G \overline{\varphi(g)}\pi(g)^*\,d\mu(g)\\ &= \int_G \overline{\varphi(g^{-1})}\pi(g)\,d\mu(g)\\ &= \int_G \varphi(g)\pi(g)\,d\mu(g) = \pi(\varphi).\end{aligned}$$

So $\pi(\varphi)$ is self-adjoint. By the Spectral Theorem for compact self-adjoint operators we decompose

$$\mathscr{H} = \mathscr{H}_0 \oplus \bigoplus_{k=1}^{\infty} \mathscr{H}_{\varphi,k}$$

where $\mathscr{H}_0$ denotes the space of functions $f \in \mathscr{H}$ of eigenvalue 0, i.e. $\pi(\varphi)f = 0$ and $\mathscr{H}_{\varphi,k}$ is the space of eigenfunction of eigenvalue λ_k.

Denote by $\mathscr{H}_*$ the minimal subspace containing all the spaces $\mathscr{H}_{\varphi,k}$ for all self-adjoint φ and $k > 0$. We claim that $\mathscr{H}_* = \mathscr{H}$.

In oder to prove the claim, assume that $\mathscr{H}_* \subsetneq \mathscr{H}$ and choose f orthogonal to $\mathscr{H}_*$ and not zero. Hence, by the decomposition, $\pi(\varphi)f = 0$. By continuity of the unitary representation we can choose for some $\varepsilon > 0$ an open neighborhood U of the identity such that

$$||\pi(g)f - f|| < \varepsilon||f||$$

for all $g \in U$. We assume in addition that φ is supported in U, only has nonnegative values and has total mass 1. Thus

$$||\pi(\varphi)f - f|| \le \int_U \varphi(g)||\pi(g)f - f||\,dg \le \varepsilon||f|| \int \varphi(g)\,d\mu(g) = \varepsilon||f||.$$

By assumption $\pi(\varphi)f = 0$ and thus the right hand side equals $||f||$, implying the relation $||f|| < \varepsilon||f||$ for all $\varepsilon > 0$. This is a contradiction. So we proved $\mathscr{H}_* = \mathscr{H}$.

Consider now $\mathscr{H}' \subset \mathscr{H}$ a non-trivial invariant subspace. Hence we can also decompose

$$\mathscr{H}' = \mathscr{H}'_0 \oplus \bigoplus_{k=1}^{\infty} \mathscr{H}'_{\varphi,k},$$

with $\mathscr{H}'_0 \subset \mathscr{H}_0$ and $\mathscr{H}'_{\varphi,k} \subset \mathscr{H}_{\varphi,k}$. Assume that $\mathscr{H}'$ has a zero intersection with all of the $\mathscr{H}_{\varphi,k}$, then we have $\mathscr{H}' \subset \mathscr{H}_*^{\perp}$. Thus $\mathscr{H}'$ is trivial. This implies that $\mathscr{H}'$ has a nonzero intersection with some $\mathscr{H}_{\varphi,k}$.

Fix for the moment $\mathscr{H}_{\varphi,k}$ and consider the set

$$\{\mathscr{H}' \cap \mathscr{H}_{\varphi,k} \,:\, \mathscr{H}' \text{ is an invariant subspace}\}.$$

In this set, choose some $\mathscr{H}'_{\varphi,k}$ nonzero of minimal dimension. Thus the set

$$\{\mathscr{H}' \subset \mathscr{H} \text{ is invariant} \,:\, \mathscr{H}' \cap \mathscr{H}_{\varphi,k} = \mathscr{H}'_{\varphi,k}\}$$

is nonempty. We denote by $\mathscr{H}_1$ the minimal of these invariant subspaces. More precisely, $\mathscr{H}_1$ is the intersection of all elements in this set.

We claim that $\mathscr{H}_1$ is irreducible. To prove this, assume that $\mathscr{H}_1$ can be decomposed into a direct sum of invariant subspaces

$$\mathscr{H}_1 = \mathscr{H}_{11} \oplus \mathscr{H}_{12}.$$

It follows that $\mathscr{H}'_{\varphi,k}$ is contained entirely in one of the subspaces $\mathscr{H}_{11}$ or $\mathscr{H}_{12}$. But this contradicts the fact that $\mathscr{H}_1$ is the minimal invariant subspace containing $\mathscr{H}'_{\varphi,k}$. So we proved the claim.

We thus can decompose

$$\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_1',$$

where $\mathscr{H}_1'$ is the orthogonal complement to $\mathscr{H}_1$. It follows that $\mathscr{H}_1'$ is invariant and has a nonzero intersection with some space $\mathscr{H}_{\varphi,k'}$. Repeating the above argument, we find in $\mathscr{H}_1'$ an invariant irreducible subspace $\mathscr{H}_2$. Since $\mathscr{H}$ is separable, we obtain a decomposition

$$\mathscr{H} = \bigoplus_{k\geq 1} \mathscr{H}_k.$$

It remains to check that the multiplicity of each invariant subspace $\mathscr{H}_k$ is finite. Let $\pi(\varphi)$ be a self-adjoint operator that has in $\mathscr{H}_k$ a nonzero eigenvalue λ. Then in every space $\mathscr{H}_l$ in which an equivalent representation acts also contains an eigenvector of $\pi(\varphi)$ with the same eigenvalue λ. Since there are only finitely many linearly independent eigenvectors of $\pi(\varphi)$ with eigenvalue λ, the number of spaces $\mathscr{H}_l$ equivalent to $\mathscr{H}_k$ is finite. Thus we can find a decomposition of $\mathscr{H}$ into a direct sum of a countable number of finite dimensional irreducible unitary representations. □

2.2 Gelfand Pairs

The main reference for this section is [Far83].

Let G be a locally compact group and let K be a compact subgroup. It will turn out that for any Gelfand pair (G, K) the group G is unimodular and so the statements from the last section apply. In order to prove unimodularity we for the moment assume that μ is a left-invariant Haar measure on G, thus for any measurable function $f : G \to \mathbb{C}$ and $h \in G$ we have

$$\int_G f(hg)\, d\mu(g) = \int_G f(g)\, d\mu(g).$$

Definition 2.5. The pair (G, K) is called a *Gelfand pair* if the set

$$C_c(G)^\natural := \{f \in C_c(G) \,:\, f(kgk') = f(g) \text{ for all } k, k' \in K, g \in G\}$$

of bi-K-invariant continuous functions of compact support is a commutative Banach algebra with respect to convolution.

We will see in the following that every Riemannian symmetric pair is a Gelfand pair. To prove this, recall that if (G, K) is a Riemannian symmetric pair, then there exists an involutive automorphism $\sigma : G \to G$, which is the identity on K. Denote

$$f^\sigma(g) := f(\sigma(g)) \qquad \text{and} \qquad f^\vee(g) := f(g^{-1})$$

and observe for $h \in G$,

$$\int_G f^\sigma(hg)\,d\mu(g) = \int_G f(\sigma(h)\sigma(g))\,d\mu(g) = \int_G f^\sigma(g)\,d\mu(g).$$

Thus, since the Haar measure is unique up to scaling, there is a constant $c > 0$ such that

$$\int_G f^\sigma(g)\,d\mu(g) = c\int_G f(g)\,d\mu(g).$$

Furthermore we have $\sigma^2 = 1$ and hence we conclude that $c^2 = 1$. Thus $c = 1$ since c is positive. So we proved

$$\int_G f^\sigma(g)\,d\mu(g) = \int_G f(g)\,d\mu(g).$$

Proposition 2.6. *Any Riemannian symmetric pair is a Gelfand pair. Furthermore, if (G, K) is a Gelfand pair, then G is unimodular.*

Proof. Let (G, K) be a Riemmanian symmetric pair and σ be the involutive automorphism, which is the identity on K. By the Cartan decomposition, every element $x \in G$ can be written as $x = kp$ with $k \in K$ and $p \in G$ such that $\sigma(k) = k$ and $\sigma(p) = p^{-1}$. Hence for $f \in C_c(G)^\natural$,

$$f(\sigma(x)) = f(\sigma(kp)) = f(k\sigma(p)) = f(\sigma(p)) = f(p^{-1}) = f(p^{-1}k^{-1}) = f(x^{-1}).$$

Thus $f^\sigma = f^\vee$. Furthermore for $f_1, f_2 \in C_c(G)^\natural$, we have

$$\begin{aligned}
(f_1^\vee * f_2^\vee)(x) &= \int_G f_1^\vee(g) f_2^\vee(g^{-1}x)\,d\mu(g)\\
&= \int_G f_1(g^{-1}) f_2(x^{-1}g)\,d\mu(g)\\
&= \int_G f_2(g') f_1(g'^{-1}x^{-1})\,d\mu(g')\\
&= (f_2 * f_1)(x^{-1})\\
&= (f_2 * f_1)^\vee(x),
\end{aligned}$$

where in the third line we used the substitution $g' = x^{-1}g$. So we proved for $f_1, f_2 \in C_c(G)^\natural$ that $f_1^\vee * f_2^\vee = (f_2 * f_1)^\vee$. In addition we derive by σ-invariance of the Haar measure,

$$\begin{aligned}
(f_1^\sigma * f_2^\sigma)(x) &= \int_G f_1(\sigma(g)) f_2(\sigma(g^{-1}x))\,d\mu(g)\\
&= \int_G f_1(g) f_2(g^{-1}\sigma(x))\,d\mu(g)\\
&= (f_1 * f_2)^\sigma(x).
\end{aligned}$$

Combining all this for $f_1, f_2 \in C_c(G)^\natural$,

$$(f_1 * f_2)^\sigma = f_1^\sigma * f_2^\sigma = f_1^\vee * f_2^\vee = (f_2 * f_1)^\vee = (f_2 * f_1)^\sigma,$$

implying $f_1 * f_2 = f_2 * f_1$. Hence (G, K) is a Gelfand pair.

Assume now that (G, K) is a Gelfand pair. Let $f_1 \in C_c(G)^\natural$. We claim $\int_G f(g^{-1})\,d\mu(g) = \int_G f(g)\,d\mu(g)$. To prove the claim consider a function $f_2 \in C_c(G)^\natural$ that is equal to 1 on the compact set $\mathrm{supp}(f_1) \cup \mathrm{supp}(f_1)^{-1}$. Since (G, K) is a Gelfand pair, we have

$$\begin{aligned}
\int_G f_1(g)\,d\mu(g) &= \int_{\mathrm{supp}(f_1)} f(g)\,d\mu(g) \\
&= \int_{\mathrm{supp}(f_1)} f(g) f_2(g^{-1})\,d\mu(g) \\
&= (f_1 * f_2)(e) \\
&= (f_2 * f_1)(e) \\
&= \int_{\mathrm{supp}(f_1)^{-1}} f_2(g) f_1(g^{-1})\,d\mu(g) \\
&= \int f_1(g^{-1})\,d\mu(g).
\end{aligned}$$

By assumption μ is left invariant. We want to show that μ is also right invariant. By the above,

$$\int_G f(gh)\,d\mu(g) = \int_G f(h^{-1}g^{-1})\,d\mu(g) = \int_G f(g^{-1})\,d\mu(g) = \int_G f(g)\,d\mu(g),$$

for all $f \in C_c(G)$ and $h \in G$. Hence G is unimodular. $\square$

We next want to decompose the space $L^2_\mu(\Gamma\backslash(G/K))$ together with the regular right representation into one-dimensional invariant and irreducible subspaces with the help of the decomposition from Theorem 2.1. The main ingredient is the following theorem.

Theorem 2.7. *Let (G, K) be a Gelfand pair and let $(\mathscr{H}, \pi)$ be an irreducible unitary representation of G. Then*

$$\dim \mathscr{H}^K \leq 1.$$

In order to prove the theorem, consider the convolution algebra $L^1_\mu(G)^\natural$ of bi-K-invariant integrable functions, where we call a function $\varphi \in L^1_\mu(G)$ *bi-K-invariant*, if for all $k, k' \in K$ and $x \in G$ we have

$$\varphi(kxk') = \varphi(x).$$

Let $\pi : G \to U(\mathscr{H})$ be a unitary representation. For $\varphi \in L^1_\mu(G)^\natural$ we note that the operator $\pi(\varphi)$ leaves the closed subspace of K-invariant elements $\mathscr{H}^K$ invariant. More precisely, if $f \in \mathscr{H}^K := \{f \in \mathscr{H} : \pi(k)f = f \text{ for all } k \in K\}$, then

$$\begin{aligned}
\pi(k)\pi(\varphi)f &= \pi(k)\int_G \varphi(g)\pi(g)\,d\mu(g)\\
&= \int_G \varphi(g)\pi(kg)\,d\mu(g)\\
&= \int_G \varphi(k^{-1}g)\pi(g)\,d\mu(g)\\
&= \int_G \varphi(g)\pi(g)\,d\mu(g)\\
&= \pi(\varphi)f.
\end{aligned}$$

Hence $\pi(\varphi)f \in \mathscr{H}^K$ and thus, we get by restriction to $\mathscr{H}^K$ a Banach algebra homomorphism

$$\pi^K : L^1_\mu(G)^\natural \to B(\mathscr{H}^K)$$

as in Lemma 2.2. The main step toward the proof of Theorem 2.7 is the following proposition.

Proposition 2.8. *If $(\mathscr{H},\pi)$ is an irreducible representation of G, then so is the representation $(\mathscr{H}^K,\pi^K)$ of $L^1_\mu(G)^\natural$.*

Proof. Let $Y \subset \mathscr{H}^K$ be a non-trivial π^K-invariant subspace and let $Y^\perp$ be its orthogonal complement in $\mathscr{H}^K$. Let f_1 be a nonzero vector of Y and denote

$$\mathscr{H}_1 = \{\pi(\varphi)f_1 \mid \varphi \in L^1_\mu(G)^\natural\}.$$

We have that $\mathscr{H}_1$ is a π^K and π-invariant subspace of $\mathscr{H}$ and hence the closure of $\mathscr{H}_1$ is equal to $\mathscr{H}$. We claim that $Y^\perp$ is orthogonal to $\mathscr{H}_1$.

Let $u_2 \in Y^\perp$ and denote by

$$\varphi^\natural(x) = \int_K\int_K \varphi(kxk')\,d\mu_K(k)d\mu_K(k'),$$

where μ_K is the normalized Haar measure on K. Hence for $\varphi \in L^1_\mu(G)^\natural$,

$$\begin{aligned}
\langle\pi(\varphi)u_1,u_2\rangle &= \int_G \varphi(g)\langle\pi(g)u_1,u_2\rangle\,d\mu(g)\\
&= \int_G \varphi(kgk')\langle\pi(g)u_1,u_2\rangle\,d\mu(g)\\
&= \int_G\int_K\int_K \varphi(kgk')\langle\pi(g)u_1,u_2\rangle\,d\mu_K(k)d\mu_K(k')d\mu(g)\\
&= \langle\pi(\varphi^\natural)u_1,u_2\rangle = 0
\end{aligned}$$

Thus $Y^\perp$ is orthogonal to $\mathscr{H}_1$. Since $\mathscr{H}_1$ is non-trivial and π-invariant, we have that $Y^\perp = 0$. We we proved that the representation $(\mathscr{H}^K,\pi^K)$ of $L^1_\mu(G)^\natural$ is irreducible. □

With the help of this proposition the theorem is a straightforward consequence.

Proof. (of Theorem 2.7) By Proposition 2.8 we have that $(\mathscr{H}^K, \pi^K, L^1_\mu(G)^\natural)$ is irreducible. The statement now follows from the following general assertion. Let $(\mathscr{H}, \pi)$ be a unitary, irreducible representation of a commutative Banach algebra, in our case $L^1_\mu(G)^\natural$. Then, if $\mathscr{H} \neq \{0\}$, then $\dim \mathscr{H} = 1$. For a proof of this fact we refer to appendix A of [Far83]. □

Corollary 2.9. *Let (G, K) be a Gelfand pair and Γ a cocompact subgroup of G. Then the space $L^2_\mu(\Gamma\backslash(G/K))$ decomposes into a countable direct sum*

$$L^2_\mu(\Gamma\backslash(G/K)) = \bigoplus_{n\geq 1} \mathscr{H}^K$$

of invariant irreducible subspaces $\mathscr{H}^K$ of dimension at most one, each of which has finite multiplicity.

Proof. By Theorem 2.1 we have a decomposition

$$L^2_\mu(\Gamma\backslash G) = \bigoplus_{n\geq 1} \mathscr{H}_n$$

where each $\mathscr{H}_n$ is an irreduicible, invariant and finite-dimensional subspace of $L^2_\mu(\Gamma\backslash G)$. Thus the corollary follows from the fact that

$$L^2_\mu(\Gamma\backslash(G/K)) = L^2_\mu(G)^K$$

together with Theorem 2.7. □

2.3 Functions of Positive Type and Spherical Functions

We follow in this section mostly [Far83].

The main objective of this section is to show that we can decompose the space $L^2_\mu(\Gamma\backslash(G/K))$, for (G, K) a Gelfand pair and Γ a cocompact discrete subgroup of G, into

$$\bigoplus_{n\geq 1} (f_n),$$

where each of the f_n generates an irreducible subspace in $L^2_\mu(\Gamma\backslash G)$. This is carried out by showing that f_n generates an irreducible subspace in $L^2_\mu(\Gamma\backslash G)$ if and only if the function $\psi(g) = \langle f, \pi(g)f\rangle$ is a so-called spherical function. Spherical functions are a special class of so called functions of positive type, which we need to study first in order to prove the desired statement. Therefore, we will more generally consider a locally compact group G.

Definition 2.10. A function $\psi : G \to \mathbb{C}$ is called of *positive type* if for all $g_1, \ldots, g_n \in G$ and $c_1, \ldots, c_n \in \mathbb{C}$,

$$\sum_{i,j=1}^{N} c_i \overline{c_j} \psi(x_j^{-1} x_i) \geq 0.$$

We further denote by $\mathscr{P}(G)^\natural$ the set of continuous bi-K-invariant functions of positive type.

Example 2.11. If $(\mathscr{H}, \pi)$ is a unitary representation of G, then for any element $u \in \mathcal{H}$ consider the function $\psi : G \to \mathbb{C}$ defined by

$$\psi(g) = \langle u, \pi(g)u \rangle.$$

These functions are of positive type since for all $g_1, \ldots, g_n \in G$ and $c_1, \ldots, c_n \in \mathbb{C}$, we have

$$\sum_{i,j=1}^{N} c_i \overline{c_j} \psi(x_j^{-1} x_i) = \left|\left| \sum_{i=1}^{n} c_i \pi(g_i) u \right|\right|^2 \geq 0.$$

An important property of functions of positive type and in particular one that we will need later is that we can associate to each such function an essentially unique unitary representation of G with a cyclic vector. This is done via the Gelfand–Naimark–Segal construction or short GNS-construction as in the next theorem.

Theorem 2.12. *(GNS-Construction) For any bi-K-invariant function $\psi : G \to \mathbb{C}$ of positive type there exists a unitary representation $(\mathscr{H}_\psi, \pi_\psi)$ of G and a cyclic vector $u \in \mathscr{H}_\psi^K$ such that*

$$\psi(x) = \langle u, \pi_\psi(x)u \rangle.$$

Furthermore the unitary representation $(\mathscr{H}_\psi, \pi_\psi)$ is unique up to isometry.

Proof. Let $\psi \in \mathscr{P}(G)^\natural$ and denote by $\mathscr{M}_0(G)$ the set of measures μ of the form

$$\mu = \sum_{i=1}^{n} a_i \delta_{g_i}$$

with $n \in \mathbb{N}$, $a_i \in \mathbb{C}$ and $g_i \in G$ for all $1 \leq i \leq n$. Additionally, we define

$$\mu * \psi(x) := \sum_{i=1}^{n} a_i \psi(g_i^{-1} x).$$

Consider the set

$$V_\psi := \{ f = \mu * \psi \, : \, \mu \in \mathscr{M}_0(G) \}.$$

We give V_ψ the structure of a pre-Hilbert space: For $\mu = \sum_{i=1}^{N} a_i \delta_{g_i}$ and $\nu = \sum_{j=1}^{M} b_j \delta_{h_j}$ we denote $f = \mu * \psi$ and $g = \nu * \psi$ and define

$$\langle f, g \rangle := \sum_{i=1}^{N} \sum_{j=1}^{M} b_i \overline{a_j} \psi(g_i^{-1} h_j).$$

This forms an inner product, since ψ is of positive type.

We consider on V_ψ the following unitary representation defined for $g \in G$ by

$$(\pi(g)f)(x) = f(g^{-1}x).$$

Observe that we can write ψ as $\psi = \delta_{e_i} * \psi$ and $\pi(g)\psi = \delta_g * \psi$. Hence

$$\langle \psi, \pi(g)\psi \rangle = \langle \delta_{e_i} * \psi, \delta_g * \psi \rangle = \psi(g).$$

Denote by $\mathscr{H}_\psi$ the completion of V_ψ, by π_ψ the extension of π onto $\mathscr{H}_\psi$ and by u the image of ψ in the extension. So for all $x \in G$,

$$\langle u, \pi_\psi(x)u\rangle = \psi(x).$$

It remains to check that u is K-invariant and cyclic. To see that u is K-invariant, note that

$$\pi(k)\psi(x) = \delta_k * \psi = \psi(k^{-1}x) = \psi(x),$$

since ψ is bi-K-invariant. We prove now that u is cyclic. Since $\pi(g)\psi = \delta_g * \psi$, we conclude

$$V_\psi = \langle \pi(g)\psi \,:\, g \in G\rangle,$$

implying that u is cyclic.

Let $(\mathscr{H}', \pi', u')$ be another such triple. Then for $\mu = \sum_{i=1}^N a_i\delta_{g_i}$, we define the linear map

$$A : V_\psi \to \mathscr{H}', \qquad\qquad f = \mu * \psi \mapsto \sum_{i=1}^N a_i\pi'(g_i)u'.$$

Thus $A\psi = u'$ and

$$\begin{aligned} ||Af||^2_{\mathscr{H}'} &= \langle Af, Af\rangle_{\mathscr{H}'} \\ &= \left\langle \sum_{i=1}^N a_i\pi'(x_i)u', \sum_{j=1}^N a_j\pi'(x_j)u' \right\rangle_{\mathscr{H}'} \\ &= \sum_{i,j=1}^n a_i\overline{a_j}\langle u', \pi'(x_i^{-1}x_j)u'\rangle_{\mathscr{H}'} \\ &= \sum_{i,j=1}^n a_i\overline{a_j}\psi(x_i^{-1}x_j) \\ &= \langle \mu*\psi, \mu*\psi\rangle_{V_\psi} = ||f||^2_{V_\psi}. \end{aligned}$$

Hence A is an isometry on V_ψ and so uniquely extends to an isometry from $\mathscr{H}_\psi$ to $\mathscr{H}'$. □

We next want to characterize the functions ψ for which $(\mathscr{H}_\psi, \pi_\psi)$ is an irreducible representation. This will later be important since we are interested in functions $f \in L^2_\mu(\Gamma\backslash(G/K))$ that generate irreducible subspaces in $L^2_\mu(\Gamma\backslash G)$. To derive a suitable condition, note that $\mathscr{P}(G)^\natural$ is a convex cone. Further, we call an element $\psi \in \mathscr{P}(G)^\natural$ *extremal* if, whenever $\psi = \psi_1 + \psi_2$ with $\psi_1, \psi_2 \in \mathscr{P}(G)^\natural$, it follows that ψ_1 and ψ_2 are proportional.

Proposition 2.13. *Let $\psi : C \to \mathbb{C}$ be a nonzero and bi-K-invariant function of positive type. Then ψ is extremal if and only if $(\mathscr{H}_\psi, \pi_\psi)$ is irreducible.*

To prove the above proposition, we will need the following version of Schur's Lemma.

Lemma 2.14. *(Schur) Let $\mathscr{H}$ be a complex Hilbert space and π be a unitary, irreducible representation. Further, consider a bounded operator $\psi : \mathscr{H} \to \mathscr{H}$. Then ψ commutes with π, i.e. $\psi \circ \pi(g) = \pi(g) \circ \psi$ for all $g \in G$ if and only if ψ is of the form $\psi = \lambda \cdot \mathrm{id}_{\mathscr{H}}$ for some $\lambda \in \mathbb{C}$.*

Proof. To prove this theorem one uses the Spectral Theorem for self-adjoint operators. For more details see section 12 of [EW17]. □

Proof. (of Proposition 2.13) Assume that ψ is extremal and that

$$\mathscr{H}_\psi = \mathscr{H}_1 \oplus \mathscr{H}_2$$

is a decomposition into closed irreducible subspaces and denote by $P_i = \mathscr{H}_\psi \to \mathscr{H}_i$ the orthogonal projection for $i = 1, 2$. Let u be such that $\psi(x) = \langle u, \pi_\psi(x)u\rangle$ and write $u = u_1 + u_2$ with $u_i = P_i u$ for $i = 1, 2$ and $\psi_i(x) = \langle u_i, \pi_\psi(x)u_i\rangle$ for $i = 1, 2$. Then

$$\psi(x) = \langle u, \pi_\psi(x)u\rangle = \langle u_1, \pi_\psi(x)u_1\rangle + \langle u_2, \pi_\psi(x)u_2\rangle = \psi_1(x) + \psi_2(x).$$

Hence, since ψ is extremal, $\psi_1 = \lambda\psi$. Next for $u = u_1 + u_2$ as above,

$$\langle u_1 - \lambda u, \pi(x)u\rangle = \langle u_1, \pi(x)u\rangle - \lambda\langle u, \pi(x)u\rangle = \psi_1(u) - \lambda\psi(u) = 0.$$

Since u is cyclic, we conclude $u_1 = \lambda u$. Hence, if $\lambda = 0$ we have $\mathscr{H}_1 = \{0\}$ and if $\lambda \neq 0$, we conclude $\mathscr{H}_1 = \mathscr{H}$. Thus $\mathscr{H}_\psi$ is irreducible.

Conversely assume $(\mathscr{H}_\psi, \pi_\psi)$ is irreducible and that $\psi = \psi_1 + \psi_2$ with $\psi_1, \psi_2 \in \mathcal{P}(G)^\natural$. For $\mu = \sum_{i=1}^N a_i\delta_{g_i}$ and $\nu = \sum_{j=1}^M b_j\delta_{h_j}$ we denote $f = \mu * \psi$ and $g = \nu * \psi$. We then define

$$B_1(f, g) = \sum_{i=1}^N \sum_{j=1}^M a_i\overline{b_j}\psi_1(h_j^{-1}g_i).$$

This is a Hermitian form with

$$|B_1(f, g)| \leq ||f||_\psi \cdot ||g||_\psi.$$

Hence the Hermitian form extends to one on $\mathscr{H}_\psi$. By the Frechet-Riesz Representation Theorem, there exists a bounded operator A on $\mathscr{H}_\psi$ such that

$$B_1(f, g) = \langle Af, g\rangle.$$

In addition, it holds for all $x \in G$,

$$B_1(\pi_\psi(x)f, \pi_\psi(x)g) = B(f, g)$$

and so for all $x \in X$

$$\pi_\psi(x) \circ A = A \circ \pi_\psi(x).$$

Then the Lemma of Schur implies that $A = \lambda\mathrm{id}_{\mathscr{H}}$ and hence $B(f, g) = \lambda\langle f, g\rangle$. Thus $\psi_1 = \lambda\psi$. □

Assume from now on that (G, K) is a Gelfand pair.

Definition 2.15. A *spherical function* is a bi-K-invariant continuous function $\psi : G \to \mathbb{C}$ such that

$$f \mapsto \chi(f) = \int_G f(g)\psi(g^{-1})\, d\mu(g)$$

is a nonzero continuous character of $C_c(G)^\natural$, i.e. for all $f_1, f_2 \in C_c(G)^\natural$ we have

$$\chi(f_1 * f_2) = \chi(f_1)\chi(f_2).$$

The next two statements characterize spherical functions.

Proposition 2.16. *Let $\psi : G \to \mathbb{C}$ be a bi-K-invariant continuous function of compact support. Then the following are equivalent:*

(i) *ψ is spherical.*

(ii) *For all $x, y \in G$*
$$\int_K \psi(xky)\, d\mu_K(k) = \psi(x)\psi(y),$$
where μ_K is the normalized Haar measure on K.

(iii) *$\psi(e) = 1$ and*
$$\psi * f = \chi(f)\psi$$
for all $f \in C_c(G)^\natural$, with $\chi(f) = \int_G f(g)\psi(g^{-1})\, d\mu(g)$.

Proof. We first show that (i) and (ii) are equivalent. The function ψ is spherical if and only if for all $f_1, f_2 \in C_c(G)^\natural$,
$$\chi(f_1 * f_2) = \chi(f_1)\chi(f_2) = \int_G \int_G f_1(g) f_2(h)\psi(g^{-1})\psi(h^{-1})\, d\mu(g)d\mu(h).$$

The left hand side is
$$\begin{aligned}
\chi(f_1 * f_2) &= \int_G (f_1 * f_2)(x)\psi(x^{-1})\, d\mu(x) \\
&= \int_G \int_G f_1(g) f_2(g^{-1}x)\psi(x^{-1})\, d\mu(g)d\mu(x) \\
&= \int_G \int_G f_1(g) f_2(h)\psi(g^{-1}h^{-1})\, d\mu(g)d\mu(h) \\
&= \int_G \int_G f_1(k^{-1}g) f_2(h)\psi(g^{-1}kh^{-1})\, d\mu(g)d\mu(h) \\
&= \int_G \int_G f_1(g) f_2(h)\psi(g^{-1}kh^{-1})\, d\mu(g)d\mu(h),
\end{aligned}$$
where we used in the third line the substitution $h = g^{-1}x$, in the following line we replaced g by $k^{-1}g$ in the next line and bi-K-invariance of f_1 in the fifth line. Hence $\chi(f_1 * f_2) - \chi(f_1)\chi(f_2) = 0$ if and only if
$$\int_G \int_G f_1(g) f_2(h) \left(\int_K \psi(g^{-1}kh^{-1})\, d\mu_K(k) - \psi(g^{-1})\psi(h^{-1}) \right) d\mu(g)d\mu(h) = 0,$$
which is equivalent to (ii).

We now prove that (ii) implies (iii). For $f \in C_c(G)^\natural$,

$$\begin{aligned}
\chi(f)\psi(x) &= \int_G f(g)\psi(g^{-1})\psi(x)\,d\mu(g) \\
&= \int_G f(g) \int_K \psi(g^{-1}kx)\,d\mu_K(k)d\mu(g) \\
&= \int_G f(k^{-1}g) \int_K \psi(g^{-1}x)\,d\mu_K(k)d\mu(g) \\
&= \int_G f(g) \int_K \psi(g^{-1}x)\,d\mu_K(k)d\mu(g) \\
&= \int_G f(g)\psi(g^{-1}x)\,d\mu(g) \\
&= (f * \psi)(x) = (\psi * f)(x),
\end{aligned}$$

where we used (ii) in the third line, then we replaced g by $k^{-1}g$ and and used bi-K-invariance in the fifth line and the assumption that (G, K) is a Gelfand pair in the last line. Further, for some $y \in G$ such that $\psi(y) \neq 0$ by bi-K-invariance,

$$\psi(e)\psi(y) = \int_K \psi(eky)\,d\mu_K(k) = \int_K \psi(ky)\,d\mu_K(k) = \int_K \psi(y)\,d\mu_K(k) = \psi(y).$$

Thus $\psi(e) = 1$. (iii) implies (ii) follows analogously. □

Theorem 2.17. *The space of bounded spherical function corresponds precisely to the space of nonzero continuous characters* $\chi : L^1_\mu(G)^\natural \to \mathbb{C}$.

Proof. If ψ is a spherical function, then the density of $C_c(G)^\natural$ in $L^1_\mu(G)^\natural$ gives rise to a character of $L^1(G)^\natural$. For the converse let χ be a nonzero continuous character of $L^1_\mu(G)^\natural$ and hence also of $C_c(G)^\natural$. By the Frechet-Riesz Representation Theorem there is a function $\psi \in C_c(G)^\natural$ with

$$\chi(f) = \int_G f(g)\psi(g^{-1})\,d\mu(g).$$

Since χ is a character, $\chi(f_1 * f_2) = \chi(f_1)\chi(f_2)$. Thus,

$$\begin{aligned}
\chi(f_1)\int \psi(g^{-1})f_2(g)\,d\mu(g) &= \chi(f_1)\chi(f_2) \\
&= \chi(f_1 * f_2) \\
&= \chi(f_2 * f_1) \\
&= \int_G (f_2 * f_1)(h)\psi(h^{-1})\,d\mu(h) \\
&= \int_G \int_G f_2(g)f_1(g^{-1}h)\psi(h^{-1})\,d\mu(g)d\mu(h) \\
&= \int_G \int_G f_1(h)\psi(h^{-1}g^{-1})f_2(g)\,d\mu(h)d\mu(g) \\
&= \int_G (\psi * f_1)(g^{-1})f_2(g)\,d\mu(g) \\
&= \int_G (f_1 * \psi)(g^{-1})f_2(g)\,d\mu(g).
\end{aligned}$$

So we proved

$$(f_1 * \psi)(x) = \chi(f_1)\psi(x).$$

Furthermore

$$\chi(f)\psi(e) = (f * \psi)(e) = \int f(g)\psi(g^{-1})\, d\mu(g).$$

So $\psi(e) = 1$. Thus ψ is spherical. $\square$

The next theorem gives a condition for when functions of positive type are spherical.

Theorem 2.18. *Let $\psi : G \to \mathbb{C}$ be a continuous bi-K-invariant function of compact support and positive type. Then ψ is spherical if and only if the unitary representation $(\mathscr{H}_\psi, \pi_\psi)$ constructed in Theorem 2.12 is irreducible and $\psi(e) = 1$.*

Lemma 2.19. *Let $(\mathscr{H}, \pi)$ be a unitary representation of G that possesses a cyclic and K-invariant vector $u \in \mathscr{H}^K$. If $\dim \mathscr{H}^K = 1$, then the representation is irreducible.*

Proof. Let $Y \subset \mathscr{H}$ be an invariant and closed subspace of $\mathscr{H}$ and let $P : \mathscr{H} \to Y$ be the projection. Observe that for all $k \in K$, $u \in \mathscr{H}$ and $v \in Y$,

$$\langle \pi(k)Pu, v\rangle = \langle Pu, \pi(k^{-1})v\rangle = \langle u, \pi(k^{-1})v\rangle = \langle \pi(k)u, v\rangle = \langle P\pi(k)u, v\rangle.$$

Thus the vector $v = Pu$ is K-invariant, since

$$\pi(k)v = \pi(k)Pu = P\pi(k)u = Pu = v,$$

where the second equals sign follows from the fact that both elements are in Y and u is cyclic. Hence $v = \lambda u$ for λ a complex number. If $\lambda = 0$, then $v = 0$ and u is orthogonal to Y. Thus $\pi(x)u$ is orthogonal. As u is cyclic we conclude $Y = \{0\}$. If $\lambda \neq 0$ we again conclude since u is cyclic that $Y = \mathscr{H}$. $\square$

Proof. (of Theorem 2.18) For convenience we will drop the index of $(\mathscr{H}_\psi, \pi_\psi)$, so we denote by $(\mathscr{H}, \pi)$ the unitary representation associated to ψ. By construction there is a K-invariant vector $u \in \mathscr{H}^K$ such that

$$\psi(x) = \langle u, \pi(x)u\rangle.$$

Assume first that ψ is spherical. In Proposition 2.16 we proved that $\psi(e) = 1$. We claim that for all $\varphi \in L^1_\mu(G)^\natural$, $\pi(\varphi)u = \chi(\varphi)u$ where $\chi(\varphi) = \int_G \varphi(g)\psi(g^{-1})\, d\mu(g)$.

To prove this claim, note that for $x \in G$ we have

$$\begin{aligned}
\langle \pi(\varphi)u, \pi(x)u\rangle &= \int_G \varphi(g)\langle \pi(g)u, \pi(x)u\rangle \, d\mu(g) \\
&= \int_G \varphi(g)\psi(g^{-1}x) \, d\mu(g) \\
&= \int_G \varphi(g)\psi(g^{-1}kx) \, d\mu(g) \\
&= \int_G \varphi(g) \int_K \psi(g^{-1}kx) \, d\mu_K(x) d\mu(g) \\
&= \int_G \varphi(g)\psi(g^{-1})\psi(x) \, d\mu(g) \\
&= \chi(\varphi)\psi(x) = \chi(\varphi)(u, \pi(x)u).
\end{aligned}$$

Hence $\pi(\varphi)u = \chi(\varphi)u$, since u is cyclic.

Next, consider the linear map

$$P : \mathscr{H} \longrightarrow \mathscr{H}^K, \qquad\qquad v \longmapsto Pv := \int_K \pi(k)v \, d\mu_K(k).$$

This is the orthogonal projection since for all $v \in \mathscr{H}$ and $v_K \in \mathscr{H}^K$

$$\begin{aligned}
\langle Pv, v_K\rangle &= \left\langle \int_K \pi(k)v \, d\mu_K(k), v_K \right\rangle \\
&= \int_K \langle \pi(k)v, v_K\rangle \, d\mu_K(k) \\
&= \int_K \langle v, \pi(k^{-1})v_K\rangle \, d\mu_K(k) \\
&= \int_K \langle v, v_K\rangle \, d\mu_K(k) \\
&= \langle v, v_K\rangle.
\end{aligned}$$

For $v = \pi(\varphi)u$ with $\varphi \in L^1_\mu(G)^\natural$,

$$\begin{aligned}
Pv = P^2 v = P\left(P\pi(\varphi)u\right) &= P\left(\int_G \int_K \pi(k)\varphi(g)\pi(g)u \, d\mu(g)\right) \\
&= P\left(\int_G \int_K \varphi(g)\pi(kg)u \, d\mu_K(k)d\mu(g)\right) \\
&= \int_G \int_K \int_K \varphi(kgk')\pi(g)u \, d\mu_K(k)d\mu_K(k')d\mu(g) \\
&= \int_G \varphi^\natural(g)\pi(g)u \, d\mu(g) \\
&= \pi(\varphi^\natural)u = \chi(\varphi^\natural)u,
\end{aligned}$$

where $\varphi^\natural(g) := \int_K \int_K \varphi(kgk') \, d\mu_K(k)d\mu_K(k')$. Since u is cyclic, this shows that the dimension of $\mathscr{H}^K$ is equal to one and thus the representation $(\mathscr{H}, \pi)$ is irreducible by Lemma 2.19.

Next, assume that $(\mathscr{H}, \pi)$ is irreducible. By Theorem 2.7, we have $\dim \mathscr{H}^K = 1$. Thus for $\varphi \in L^1_\mu(G)^\natural$, we conclude

$$\pi(\varphi)u = \chi(\varphi)u$$

for some scalar $\chi(\varphi) \in \mathbb{C}$. Thus for all $\varphi_1, \varphi_2 \in L^1_\mu(G)^\natural$,

$$\chi(\varphi_1 * \varphi_2)u = \pi(\varphi_1 * \varphi_2)u = \pi(\varphi_1)\pi(\varphi_2)u = \chi(\varphi_1)\chi(\varphi_2)u.$$

So χ is a character. Since $\psi(e) = 1$,

$$\begin{aligned}\chi(\varphi) &= \chi(\varphi)\psi(e) \\ &= \langle \pi(\varphi)u, \pi(e)u\rangle \\ &= \int_G \varphi(g)\langle \pi(g)u, \pi(e)u\rangle \, d\mu(g) \\ &= \int_G \varphi(g)\psi(g^{-1}) \, d\mu(g).\end{aligned}$$

Hence $\chi(\varphi) = \int_G \varphi(g)\psi(g^{-1})\, d\mu(g)$ is a character and thus ψ is spherical. □

Combining all this we can finally prove the following two corollaries, fulfilling the aim stated in the beginning of this section.

Corollary 2.20. *Let $\Gamma\backslash G$ be compact and $f \in L^2_\mu(\Gamma\backslash(G/K))$. Then the following properties are equivalent.*

(i) f generates an irreducible subspace in $L^2_\mu(\Gamma\backslash G)$, i.e. $\overline{\langle \pi(g)f \,:\, g \in G\rangle} \subset L^2_\mu(\Gamma\backslash G)$ is irreducible.

(ii) For all $\varphi \in L^1_\mu(G)^\natural$ we have

$$\pi(\varphi)f = \chi(\varphi)f,$$

where $\chi(\varphi) = \int_G \varphi(g)\psi(g^{-1})\, d\mu(g)$ and $\psi = \langle f, \pi(g)f\rangle$.

(iii) The function

$$\psi(g) = \langle f, \pi(g)f\rangle$$

for $g \in G$ is spherical.

Proof. Note that in (i) we view f as a function from $\Gamma\backslash G \to \mathbb{C}$. Furthermore the subspace $\mathscr{H} := \overline{\langle \pi(g)f \,:\, g \in G\rangle}$ is invariant under π. Hence we can view $(\mathscr{H}, \pi)$ as the, up to isometry, unique unitary representation associated to ψ by the GNS-construction in Theorem 2.12 with cyclic vector f such that $\psi(g) = \langle f, \pi(g)f\rangle$. Thus (i) and (iii) are equivalent by Theorem 2.18.

The statements (i) implies (ii) and (ii) implies (iii) are proved as in Theorem 2.18. □

Corollary 2.21. *Let (G, K) be a Gelfand pair and Γ a cocompact discrete subgroup G. Then*

$$L^2_\mu(\Gamma\backslash(G/K)) = \bigoplus_{n\geq 1}(f_n)$$

where each f_n generates an irreducible subspace of $L^2_\mu(\Gamma\backslash G)$.

Proof. Denote $\mathscr{H} := L^2_\mu(\Gamma\backslash G)$, hence $\mathscr{H}^K = L^2_\mu(\Gamma\backslash(G/K))$. By Theorem 2.1 we get a decomposition

$$\mathscr{H} = \bigoplus_{n\geq 1} \mathscr{H}_n$$

with $\mathscr{H}_n$ finite dimensional irreducible subspaces of finite multiplicity. Then we have as in Corollary 2.9

$$\mathscr{H}^K = \bigoplus_{n\geq 1} \mathscr{H}_n^K$$

where for each n we have $\dim \mathscr{H}_n^K \leq 1$. Thus we can omit those n for which $\mathscr{H}_n^K = \{0\}$ and hence can choose for every n a function $f_n \in \mathscr{H}_n^K$ such that $(f_n) := \mathbb{C} \cdot f_n = \mathscr{H}_n^K$. So

$$\mathscr{H}^K = \bigoplus_{n\geq 1} (f_n),$$

where each f_n appears only finitely often. Furthermore the subspace

$$\overline{\langle \pi(g) f_n \,:\, g \in G \rangle} \subset \mathscr{H}_n$$

is non-trivial and invariant under π. Thus, since $\mathscr{H}_n$ is irreducible, we conclude

$$\overline{\langle \pi(g) f_n \,:\, g \in G \rangle} = \mathscr{H}_n.$$

This implies that this space is irreducible since $\mathscr{H}_n$ is irreducible. Hence f_n generates an irreducible subspace. ☐

2.4 The Laplace Operator and Point Pair Invariant Functions

For the rest of this thesis we reduce to the case $G = \mathrm{PSL}_2(\mathbb{R})$ and $K = \mathrm{PSO}_2(\mathbb{R})$, hence $G/K = \mathbb{H}$. This is a Riemannian symmetric pair and thus the theory developed in the preceding sections applies. More precisely, we proved for the compact hyperbolic surface $S := \Gamma\backslash\mathbb{H}$,

$$L^2_\mu(S) = \bigoplus_{n\geq 1} (f_n) \tag{2.1}$$

where each f_n generates an irreducible subspace in $L^2_\mu(\Gamma\backslash G)$. We will show in this section that these functions are precisely eigenfunctions for the Laplace operator.

We use the following terminology. For $\mathscr{H}$ a Hilbert space, we call any linear map defined on a dense subset of $\mathscr{H}$ an *operator*. The space where the operator is defined is called the *domain*. Further, one calls an operator $T : \mathscr{H} \to \mathscr{H}$ *symmetric* if

$$\langle Tf, g \rangle = \langle f, Tg \rangle.$$

The Laplace operator on $\mathbb{H}$ is given by

$$\triangle = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

This operators acts on $C^\infty(\mathbb{H})$. We want to see that the Laplace operator is symmetric on the space of smooth functions with compact support. In order to prove this, we use the following notations. First denote by

$$\triangle^e = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}$$

the usual euclidean Laplacian and further by

$$\nabla f := \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

the gradient.

Proposition 2.22. *The Laplace operator is a symmetric operator on $L^2_\mu(\mathbb{H})$ with domain $C_c^\infty(\mathbb{H})$.*

Proof. Let $f, g \in C_c^\infty(\mathbb{H})$ be two smooth functions of compact support. Then we have

$$d\left(g\left(\frac{\partial f}{\partial x}dy - \frac{\partial f}{\partial y}dx\right)\right) = (g\triangle^e f + \nabla f \cdot \nabla g)\, dx \wedge dy \tag{2.2}$$

and hence

$$d\left(g\left(\frac{\partial f}{\partial x}dy - \frac{\partial f}{\partial y}dx\right) - f\left(\frac{\partial g}{\partial x}dy - \frac{\partial g}{\partial y}dx\right)\right) = (g\triangle^e f - f\triangle^e g)\, dx \wedge dy.$$

Hence, by Stokes' theorem

$$\int_D (g\triangle^e f - f\triangle^e g)\, dx \wedge dy = \int_{\partial D} \left(g\left(\frac{\partial f}{\partial x}dy - \frac{\partial f}{\partial y}dx\right) - f\left(\frac{\partial g}{\partial x}dy - \frac{\partial g}{\partial y}dx\right)\right)$$

where we consider a region D that contains the support of both f and g. Hence the integral on the right hand side is zero and so we conclude

$$\int_{\mathbb{H}} (\overline{g}\triangle^e f - f\triangle^e \overline{g})\, dx \wedge dy = 0$$

or equivalently,

$$\int_{\mathbb{H}} \overline{g}(\triangle^e f)\, dx \wedge dy = \int_{\mathbb{H}} f(\triangle^e \overline{g})\, dx \wedge dy. \tag{2.3}$$

The fact that the operator is symmetric now follows from (2.3) since

$$\langle \triangle f, g\rangle = \int_{\mathbb{H}} (\triangle f)\overline{g}\, \frac{dx \wedge dy}{y^2} = -\int_{\mathbb{H}} \overline{g}(\triangle^e f)\, dx \wedge dy.$$

□

In order to see that the Laplace operator defines an operator on $C^\infty(S)$, we show that the Laplace operator is invariant by the action of $\mathrm{PSL}_2(\mathbb{R})$ on $C^\infty(\mathbb{H})$. This is proved in the next proposition.

Proposition 2.23. *Consider the action of* $\mathrm{PSL}_2(\mathbb{R})$ *on* $C^\infty(\mathbb{H})$ *given, for* $\gamma \in \mathrm{PSL}_2(\mathbb{R})$*, by*

$$\gamma : C^\infty(\mathbb{H}) \to C^\infty(\mathbb{H}), \qquad f \mapsto f \circ \gamma.$$

Then the Laplace operator on $\mathbb{H}$ *is invariant under this action.*

Proof. It suffices to check the statement for the generators of $\mathrm{PSL}_2(\mathbb{R})$ which are by Lemma 1.5

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with $t \in \mathbb{R}$ and $s \in \mathbb{R}\backslash\{0\}$. For $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ of the first type, we observe for $f \in C^\infty(\mathbb{H})$

$$\triangle(f(\gamma z)) = \triangle(f(z+t)) = (\triangle f)(z+t).$$

Next, let $\gamma = \left(\begin{smallmatrix} s & 0 \\ 0 & s^{-1} \end{smallmatrix}\right)$ for $s \in \mathbb{R}\backslash\{0\}$, then

$$\begin{aligned}\triangle(f(\gamma z)) &= \triangle(f(s^2 z)) \\ &= -(s^2 y)^2 \left(\frac{\partial^2}{\partial (sx)^2} + \frac{\partial^2}{\partial (sy)^2}\right) f(s^2 z) \\ &= (\triangle f)(s^2 z).\end{aligned}$$

We leave it as an exercise to check invariance for the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. $\square$

So if $f \in C^\infty(S)$, we can view the function f as a Γ-invariant element of $C^\infty(\mathbb{H})$. If we then apply the Laplace operator to f, we get a function on $C^\infty(\mathbb{H})$ which is again Γ-invariant. Thus the Laplace operator is well defined on S. Denote further by

$$\mathcal{D}(S) = \{f \in C^\infty(S) \,:\, f \text{ bounded and } \triangle f \text{ bounded}\}.$$

Proposition 2.24. *The Laplacian is a symmetric positive operator on* $L^2_\mu(S)$ *with domain* $\mathcal{D}(S)$

Proof. We follow [Ber11] section 3.6. Let f, g be two functions contained in $\mathcal{D}(S)$. We consider the differential form $\omega = g\left(\frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx\right)$. Further consider the 2-form

$$df \wedge \left(\frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx\right) = |y \nabla f|^2 \, \frac{dx \wedge dy}{y^2},$$

which is Γ invariant (See Lemma 3.15 of [Ber11]). Thus with Stokes' theorem, since a compact hyperbolic surface has no boundary and with (2.2), we derive

$$\langle \triangle f, g \rangle = -\int_D \overline{g} \triangle^e f \, dx \wedge dy = \int_D (\nabla \overline{g})(\nabla f) \, dx \wedge dy,$$

where D is a fundamental domain for Γ. Thus the operator is symmetric and it is positive since

$$\langle \triangle f, f \rangle = \int_D |\nabla f|^2 \, dx \wedge dy \geq 0. \tag{2.4}$$

$\square$

Corollary 2.25. *The eigenvalues of the Laplace operator on a compact hyperbolic surface are positive real numbers.*

Proof. Since the operator is symmetric, we conclude from $\triangle f = \lambda f$ for a nonzero function f, that

$$\lambda\langle f, f\rangle = \langle \lambda f, f\rangle = \langle \triangle f, f\rangle = \langle f, \triangle f\rangle = \langle f, \lambda f\rangle = \overline{\lambda}\langle f, f\rangle$$

and so λ is a real number. To see that the eigenvalues are positive, we use equation (2.4) to see $\lambda\langle f, f\rangle = \langle \triangle f, f\rangle = \int_D |\nabla f|^2 \, dx \wedge dy \geq 0$ and hence $\lambda \geq 0$. $\square$

Via the decomposition (2.1) and Corollary 2.20 we are interested in functions $f \in L^2_\mu(S)$ such that $\pi(\varphi)f_n = \chi(\varphi)f_n$ for all $\varphi \in L^1_\mu(G)^\natural$. A class of functions that is closely related to functions in $L^1_\mu(G)^\natural$ are so-called point pair invariant functions. We next define them and then explain how they relate the functions in $L^1_\mu(G)^\natural$.

Definition 2.26. A smooth function $k : \mathbb{H}\times\mathbb{H} \to \mathbb{R}$ is called *point pair invariant* if $k(z, w)$ depends only on the distance between z and w. Hence there exits an even function $k_\mathbb{R} : \mathbb{R} \to \mathbb{R}$ such that

$$k_\mathbb{R}(d(z, w)) := k(z, w).$$

We call the point pair invariant function k of *compact support* if $k_\mathbb{R}$ is of compact support.

Let $\varphi \in L^1_\mu(G)^\natural$. Since $\mathbb{H} = G/K$, there is a function $\varphi_\mathbb{H} \in L^1_\mu(\mathbb{H})$ such that

$$\varphi_\mathbb{H}(z) = \varphi(z),$$

where we view z as an element of $\mathbb{H}$ on the left-hand side and z as an element of G on the right hand side. Furthermore $\varphi_\mathbb{H}$ is right-K-invariant and thus $\varphi_\mathbb{H}(z)$ only depends on the distance of z to $i \in \mathbb{H}$. If $\varphi_\mathbb{H}$ is additionally smooth, there consequently exists a unique point pair invariant function $k : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ such that

$$k(z, i) = \varphi_\mathbb{H}(z).$$

The uniqueness of such a point pair invariant function k follows since the action of G on $\mathbb{H}$ is transitive. In this setting note further for $f \in L^2_\mu(\mathbb{H}) = L^2_\mu(G/K)$ and μ the Haar measure of G,

$$\begin{aligned}
(\pi(\varphi)f)(x) &= \int_G \varphi(g)f(xg)\, d\mu(g) \\
&= \int_G \varphi(x^{-1}g)f(g)\, d\mu(g) \\
&= \int_K \int_\mathbb{H} \varphi(x^{-1}zk)f(zk)\, d\mu_\mathbb{H}(z)d\mu(k) \\
&= \int_\mathbb{H} \varphi(x^{-1}z)f(z)\, d\mu_\mathbb{H}(z) \\
&= \int_\mathbb{H} k(i, x^{-1}z)f(z)\, d\mu_\mathbb{H}(z) \\
&= \int_\mathbb{H} k(x, z)f(z)\, d\mu_\mathbb{H}(z).
\end{aligned}$$

So if we define an operator $\pi(k)$ as

$$\pi(k)f = \int_{\mathbb{H}} k(x,z)f(z)\,d\mu_{\mathbb{H}}(z)$$

we hence have

$$\pi(\varphi)f = \pi(k)f.$$

In analogy to $L^1_\mu(G)^\natural$, we consider the commutative Banach algebra

$$\mathscr{B} := \left\{\text{point pair invariant } k : \mathbb{H}\times\mathbb{H} \to \mathbb{C} \text{ with } \int_{\mathbb{H}} |k(z,i)|\,d\mu_{\mathbb{H}}(z) < \infty\right\}.$$

Thus the characters $\chi : L^1_\mu(G) \to \mathbb{C}$ correspond precisely to characters $\chi : \mathscr{B} \to \mathbb{C}$. Using the decomposition (2.1) we arrive at the following corollary.

Corollary 2.27. *For $\chi : \mathscr{B} \to \mathbb{C}$ a nontrivial continuous character, denote by*

$$E_\chi := \{f \in L^2_\mu(S), \pi(k)f = \chi(k)f \text{ for all } k \in \mathscr{B}\}.$$

Then we have a decomposition

$$L^2_\mu(S) = \bigoplus_\chi E_\chi,$$

where the direct sum is taken over all non-trivial continuous characters χ and for all χ, $\dim E_\chi < \infty$.

Proof. This is precisely the decomposition (2.1) together with Corollary 2.20, where we group the functions together according to the character χ such that $\pi(k)f = \chi(k)f$ for all $k \in \mathscr{B}$. It remains to argue that each of the spaces E_χ are finite dimensional. If two functions f_1 and f_2 in (2.1) are eigenspaces of the same character χ then by Corollary 2.20 the associated spherical functions are also the same. Thus the irreducible subspaces $\mathscr{H}_1$ and $\mathscr{H}_2$ generated by f_1 and f_2 are isometric. In Corollary 2.9 we have seen that each of these spaces occur only with finite multiplicity and hence E_χ is finite dimensional. □

Point pair invariant functions have useful properties with respect to the Laplace operator as the next two statements show.

Lemma 2.28. *Let $k(z,w)$ be a point pair invariant function. Then*

$$\triangle_z k(z,w) = \triangle_w k(z,w).$$

Furthermore, both of the functions $\triangle_z k(z,w)$ and $\triangle_w k(z,w)$ are point pair invariant.

Proof. In polar coordinates about the point w, we derive by point pair invariance

$$\triangle_z k(z,w) = -k''(r) - \coth(r)k'(r).$$

One obtains the same formula if one considers $k(z,w)$ in polar coordinates about the point z. □

Proposition 2.29. *The operators $\pi(k)$ commute with the action of the Laplacian.*

Proof. Since the Laplace operator is symmetric, we obtain

$$\int_{\mathbb{H}} k(z,w)(\triangle_w f(w))\, d\mu_{\mathbb{H}}(w) = \int_{\mathbb{H}} (\triangle_w k(z,w)) f(w)\, d\mu_{\mathbb{H}}(w).$$

Then by the last lemma the integral on the right is equal to

$$\int_{\mathbb{H}} (\triangle_z k(z,w)) f(w)\, d\mu_{\mathbb{H}}(w).$$

□

We are now ready to prove the main theorem of this section.

Theorem 2.30. *A function $f \in L^2_\mu(\Gamma\backslash\mathbb{H})$ generates an irreducible subspace in $L^2_\mu(\Gamma\backslash G)$ if and only if f is an eigenfunction of the Laplace operator. Furthermore, the eigenspaces of the Laplace operator are finite dimensional.*

Proof. Assume that f generates an irreducible subspace in $L^2_\mu(\Gamma\backslash G)$, then by Corollary 2.20 we have that f is contained in E_χ for some non-trivial continuous character χ. Hence

$$\chi(k)f(x) = (\pi(k)f)(x) = \int_{\mathbb{H}} k(x,y)f(y)\, d\mu_{\mathbb{H}}(y)$$

for all point pair invariant $k \in \mathscr{B}$ and so f is smooth. By Lemma 2.28 we further conclude for all $k \in \mathscr{B}$,

$$\begin{aligned}
\chi(k)\triangle_x f(x) &= \triangle_x(\pi(k)f)(x) \\
&= \triangle_x \int_{\mathbb{H}} k(x,y)f(y)\, d\mu_{\mathbb{H}}(y) \\
&= \int_{\mathbb{H}} (\triangle_x k(x,y))f(y)\, d\mu_{\mathbb{H}}(y) \quad = (\pi(\triangle_x k)f)(x) = \chi(\triangle_x k)f(x).
\end{aligned}$$

So if we choose a function $k \in \mathscr{B}$ such that $\chi(k) \neq 0$, we have just proved that f is an eigenfunction of the Laplacian with eigenvalue $\frac{\chi(\triangle_x k)}{\chi(k)}$. Hence $E_\chi \subset E_\lambda$, where E_λ denotes the eigenspace of the Laplace operator with eigenvalue λ. If $\lambda_1 \neq \lambda_2$, then the eigenspaces E_{λ_1} and E_{λ_2} are orthogonal to each other, since for $f \in E_{\lambda_1}$ and $g \in E_{\lambda_2}$, we have

$$\lambda_1\langle f,g\rangle = \langle \lambda_1 f, g\rangle = \langle \triangle f, g\rangle = \langle f, \triangle g\rangle = \langle f, \lambda_2 g\rangle = \lambda_2\langle f,g\rangle,$$

where we used that the eigenvalues are real numbers. So we conclude $\langle f,g\rangle = 0$.

By Corollary 2.27 we see that the spaces E_χ are orthogonal for distinct characters. Thus $E_\chi = E_\lambda$. □

Corollary 2.31. *For any compact hyperbolic surface $S = \Gamma\backslash\mathbb{H}$ there exists a complete orthonormal system of eigenfunctions of the Laplace operator. Furthermore each eigenvalue occurs with finite multiplicity.*

Proof. This is (2.1) together with Theorem 2.30. □

The next two statements, which we won't prove, state some more properties of the eigenvalues of the Laplace operator. For a proof of these two statements in a closely related setting we refer to [Bum98] pages 176 to 185. One can also find a proof of these statements in chapter 3 of [Ber11] or in chapter 7 of [Bus92].

Proposition 2.32. *The eigenspace of the eigenvalue* 0 *for a compact hyperbolic surface is one dimensional.*

Theorem 2.33. *Let Let* $(\lambda_i)_{i=1}^{\infty}$ *be the eigenvalues of the Laplace operator on a compact hyperbolic surface. Then*

$$\sum_{i=0}^{\infty} \lambda_i^{-2} < \infty.$$

From this theorem we derive that the sequence of eigenvalues converges to infinity and hence we can order them increasingly. Throughout the rest of this thesis we count the eigenvalues $(\lambda_i)_{i=0}^{\infty}$ with multiplicity and assume that they are ordered as follows

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots .$$

3 The Selberg Trace Formula and Counting of Closed Geodesics

In this section we intend to prove the Selberg trace formula and the prime geodesic theorem. In order to accomplish this, we will investigate the so-called *invariant integral operators* $\pi(k)$ from section 2.4 more closely. However, to avoid any convergence issues, we will assume that the point-pair invariant function k is of compact support. It turns out, as we will prove in section 3.1, that these operators have the same eigenfunctions as the Laplacian and the eigenvalues are also in a direct relation with the eigenvalues of the Laplacian.

In section 3.2 we proceed by considering the invariant integral operators $\pi(k)$ on a compact hyperbolic surface S. In this case, using the convergence properties of the eigenvalues of the Laplacian from the end of the last section, we can show that the operators $\pi(k)$ are of trace class, i.e the sum of its eigenvalues converges. Furthermore, the operators $\pi(k)$ are also integral. This is particularly interesting as the trace of integral operators can be expressed in a useful way. Namely, the trace of $\pi(k)$ is equal to

$$\int k(x,x)\, d\mu(x).$$

The Selberg trace formula will be derived from this expression of the trace of $\pi(k)$ by using intricate transformations. We then prove the Selberg trace formula for a wider class of functions than just the ones of compact support.

In section 3.3 we prove the Weyl law, a description of the asymptotic behavior of the number of eigenvalues less than a constant. The Selberg trace formula is used to prove the Weyl law.

Finally, in section 3.4 we prove the prime geodesic theorem, a precise estimate of the number for prime geodesics. We prove the prime geodesic theorem by a intricate use of the Selberg trace formula. The prime geodesic theorem then implies the goal of this thesis, namely to show that the number of closed geodesics with length less than any fixed number L behaves asymptotically as

$$\frac{e^L}{L}$$

as L tends to infinity.

The main references for this section are [Ber11] and [Bus92].

3.1 Eigenvalues of Invariant Integral Operators

This section closely follows [Ber11] sections 3.3 and 3.4.

We want to study the eigenvalues of the *invariant integral operators* $\pi(k)$ which we already considered in section 2.4. However, we restrict to the case where k is of compact support. We first give a more precise definition of the invariant integral operators.

Definition 3.1. Let $k : \mathbb{H}\times\mathbb{H} \to \mathbb{R}$ be a point pair invariant function of compact support. We define the *invariant integral operator* associated to k as

$$\pi(k) : L^2_\mu(\mathbb{H}) \to L^2_\mu(\mathbb{H}), \qquad f \mapsto \int_{\mathbb{H}} k(\cdot, w) f(w)\, d\mu_{\mathbb{H}}(w).$$

The main and only theorem of this section shows that every eigenfunction of the Laplacian is also an eigenfunction of the invariant integral operator and the eigenvalues are also closely related. This is an important step towards the Selberg trace formula.

Theorem 3.2. *Let $k : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be a compactly supported smooth point pair invariant function. Then there exists a function $h : \mathbb{C} \to \mathbb{C}$ such that if $f \in C^\infty(\mathbb{H})$ is an eigenfunction of the Laplacian with eigenvalue λ, then*

$$(\pi(k)f)(z) = \int_{\mathbb{H}} k(z,w) f(w)\, d\mu_{\mathbb{H}}(w) = h(\lambda) f(z). \tag{3.1}$$

More concretely, h is given, for r a positive or positive imaginary solution of $r^2 + \frac{1}{4} = \lambda$, by

$$\begin{aligned} h(r) &= \int_0^\infty \int_{-\infty}^\infty U\left(\frac{1+x^2+y^2}{2y}\right) y^{\frac{1}{2}+ir} \frac{dxdy}{y^2} &(3.2)\\ &= \sqrt{2}\int_{-\infty}^\infty e^{iru} \int_{|u|}^\infty \frac{k_{\mathbb{R}}(\rho)\sinh(\rho)}{\sqrt{\cosh(\rho)-\cosh(u)}}\, d\rho du &(3.3)\end{aligned}$$

where $U(\cosh d) = k_{\mathbb{R}}(d)$.

To prove the theorem we will construct a radial function with useful properties, where we note that we call a function $\varphi : \mathbb{H} \to \mathbb{C}$ radial about $z \in \mathbb{H}$ if $\varphi(x)$ depends only on the distance between x and z. We furthermore remark that for any $\lambda \in \mathbb{C}$ and $z \in \mathbb{H}$ there exists a unique eigenfunction of the Laplacian $\varphi_\lambda(z,w)$ with eigenvalue λ which is radial about z and satisfies $\varphi_\lambda(z,z) = 1$. For more details we refer to [Ber11] Lemma 3.6.

Proof. Let $f \in C^\infty(\mathbb{H})$ be an eigenfunction of the Laplacian with eigenvalue λ. Denote further by

$$f_z^{\mathrm{rad}}(w) := \int_{\mathrm{Stab(z)}} f(Tw)\, dT$$

for dT the normalized Haar measure on $\mathrm{Stab}(z)$. We remark that $\mathrm{Stab}(z)$ is conjugated to $\mathrm{SO}_2(\mathbb{R})$ and hence can be identified with $\mathrm{SO}_2(\mathbb{R})$. Hence dT can be viewed as the Lebesgue measure. Furthermore note that $f_z^{\mathrm{rad}}(w)$ is radial around z and satisfies $f_z^{\mathrm{rad}}(z) = f(z)$. Hence $f_z^{\mathrm{rad}}(w) = \varphi_\lambda(z,w) f(z)$ for $\varphi_\lambda(z,w)$ the unique normalized radial function about z that is an eigenfunction of the Laplacian with eigenvalue λ. Observe

$$\begin{aligned}\int_{\mathbb{H}} f_z^{\mathrm{rad}}(w) k(z,w)\, d\mu_{\mathbb{H}}(w) &= \int_{\mathbb{H}} k(z,w) \int_{\mathrm{Stab}(z)} f(Tw)\, dT d\mu_{\mathbb{H}}(k)\\ &= \int_{\mathrm{Stab}(z)} \int_{\mathbb{H}} k(z,w) f(Tw)\, d\mu_{\mathbb{H}}(w) dT\\ &= \int_{\mathrm{Stab}(z)} \int_{\mathbb{H}} k(z,T^{-1}w) f(w)\, d\mu_{\mathbb{H}}(w) dT\\ &= \int_{\mathrm{Stab}(z)} \int_{\mathbb{H}} k(z,w) f(w)\, d\mu_{\mathbb{H}}(w) dT\\ &= \int_{\mathbb{H}} k(z,w) f(w)\, d\mu_{\mathbb{H}}(w).\end{aligned}$$

Hence

$$\begin{aligned}\int_{\mathbb{H}} k(z,w)f(w)\,d\mu_{\mathbb{H}}(w) &= \int_{\mathbb{H}} k(z,w)f_z^{\mathrm{rad}}(w)\,d\mu_{\mathbb{H}}(w)\\ &= f(z)\int_{\mathbb{H}} k(z,w)\varphi_\lambda(z,w)\,d\mu_{\mathbb{H}}(w).\end{aligned}$$

This term $\int_{\mathbb{H}} k(z,w)\varphi_\lambda(z,w)\,d\mu_{\mathbb{H}}(w)$ does not depend on z, as the following calculation shows together with the transitivity of the group action

$$\begin{aligned}\int_{\mathbb{H}} k(gz,w)\varphi_\lambda(gz,w)\,d\mu_{\mathbb{H}}(w) &= \int_{\mathbb{H}} k(z,g^{-1}w)\varphi_\lambda(z,g^{-1}w)\,d\mu_{\mathbb{H}}(w)\\ &= \int_{\mathbb{H}} k(z,w)\varphi_\lambda(z,w)\,d\mu_{\mathbb{H}}(w).\end{aligned}$$

We thus define

$$h(\lambda) = \int_{\mathbb{H}} k(z,w)\varphi_\lambda(z,w)\,d\mu_{\mathbb{H}}(w)$$

for any $z \in \mathbb{H}$. This shows (3.1).

For the explicit calculation of h, note that $y^{\frac{1}{2}+ir}$ for $r \in \mathbb{C}$ for $y = \operatorname{Im} z$ is an eigenfunction of the Laplacian on $\mathbb{H}$ with eigenvalue $r^2 + \frac{1}{4}$ and furthermore $f(i) = 1$. Hence by (3.1)

$$h\left(r^2 + \frac{1}{4}\right) = \int_{\mathbb{H}} k(i,z)y^{\frac{1}{2}+ir}\,d\mu_{\mathbb{H}}(z).$$

For $U(\cosh\, d) = k_{\mathbb{R}}(d)$ together with (1.4) note that

$$k(z,i) = U(\cosh(d(i,z))) = U\left(1 + \frac{|z-i|^2}{2y}\right) = U\left(\frac{1+x^2+y^2}{2y}\right)$$

for $z = x+iy$. So equation (3.2) follows. To deduce equation (3.3) one carries out three substitutions: $t = \frac{1+x^2+y^2}{2y}$, $y = e^u$ and $t = \cosh(\rho)$. $\square$

Abbreviate

$$g(u) := \sqrt{2}\int_{|u|}^{+\infty} \frac{k_{\mathbb{R}}(\rho)\sinh(\rho)}{\sqrt{\cosh(\rho)-\cosh(u)}}\,d\rho$$

and note that h is equal to the Fourier transform $\int_{\infty}^{\infty} e^{iru}g(u)\,du$ of g. If we next use the two substitutions $x = \cosh(u)$ and $t = \cosh(\rho)$, we arrive at the expression

$$g(\operatorname{arcch}(x)) = \sqrt{2}\int_x^{\infty} \frac{k_{\mathbb{R}}(\operatorname{arcch}(t))}{\sqrt{t-x}}\,dt.$$

Hence $g(\operatorname{arcch}(x))$ is almost the *Abel transform* of the function $k(\operatorname{arcch}(t))$ as the next definition shows.

Definition 3.3. Let $f : [1,\infty) \to \mathbb{C}$ be a compactly supported smooth function. Then the *Abel transform* of f defined for $x \geq 1$ is given by

$$A[f](x) = \int_x^{\infty} \frac{f(t)}{\sqrt{t-x}}\,dx.$$

The next proposition states some properties of the Abel transform.

Proposition 3.4. *Let $f : [1, \infty) \to \mathbb{C}$ be a compactly supported smooth function. Then the Abel transform $g = A[f]$ is also compactly supported and smooth. Furthermore, the Abel transform is invertible with inverse given for $y \geq 1$ by*

$$A^{-1}[g](y) = -\frac{1}{\pi} \int_x^\infty \frac{dg(t)}{\sqrt{t-x}}.$$

Proof. Observe by substituting $t = t'^2 + x$,

$$g(x) = \int_x^\infty \frac{f(t)}{\sqrt{t-x}}\, dt = 2 \int_0^\infty f(x + t'^2)\, dt'$$

and hence the Abel transform is compactly supported and smooth with derivative

$$g'(x) = 2 \int_0^\infty f'(x + t'^2)\, dt'.$$

Thus we conclude

$$\begin{aligned}
f(y) &= -\int_0^\infty 2f'(y + r^2) r\, dr \\
&= -\frac{4}{\pi} \int_0^{\frac{\pi}{2}} \int_0^\infty f'(x + r^2) r\, dr d\theta \\
&= -\frac{4}{\pi} \int_0^\infty \int_0^\infty f'(x + \eta^2 + \xi^2)\, d\eta d\xi \\
&= -\frac{2}{\pi} \int_0^\infty g'(x + \eta^2)\, d\eta \\
&= -\frac{1}{\pi} \int_0^\infty \frac{d[g(x + \eta^2)]}{\eta} \\
&= -\frac{1}{\pi} \int_x^\infty \frac{dg(t)}{\sqrt{t-x}}.
\end{aligned}$$

□

3.2 The Selberg Trace Formula

Consider a compact hyperbolic surface $S = \Gamma\backslash\mathbb{H}$. A function $f : S \to \mathbb{C}$ can also be viewed as a Γ-invariant function on $\mathbb{H}$. So we can consider the invariant integral operators for S defined for a compactly supported smooth function $k : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ by

$$\pi_S(k) : L^2_\mu(S) \to L^2_\mu(S), \qquad f \mapsto \int_{\mathbb{H}} k(\cdot, w) f(w)\, d\mu_{\mathbb{H}}(w).$$

One straightforwardly checks that $\pi_S(k)f$ is indeed contained in $L^2_\mu(S)$. The invariant integral operators for S have interesting properties that will allow us to relate the eigenvalues of the Laplacian to the length of closed geodesics. First observe, as in the proof of Lemma 2.3, that

$$\begin{aligned}
\int_{\mathbb{H}} k(z, w) f(w)\, d\mu_{\mathbb{H}}(w) &= \sum_{\gamma \in \Gamma} \int_D k(z, \gamma w) f(w)\, d\mu_{\mathbb{H}}(w) \\
&= \int_D \sum_{\gamma \in \Gamma} k(z, \gamma w) f(w)\, d\mu_{\mathbb{H}}(w),
\end{aligned}$$

where D is a fundamental domain for S. As in Lemma 2.3, note that the *Selberg kernel*

$$K(z,w) := \sum_{\gamma\in\Gamma} k(z,\gamma w)$$

is a smooth function since the function k is smooth and we only sum over a finite number of elements $\gamma \in \Gamma$ as k has compact support. Hence the operator $\pi_S(k)$ is compact by Theorem B.28 in the appendix.

Consider next the ordered eigenvalues of the Laplace operator $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ where we count the eigenvalues with multiplicity. Write $f_0, f_1, f_2, \dots$ for a complete orthonormal system of real eigenfunctions where f_i has eigenvalue λ_i. Since we can view f_i as an eigenfunction of the Laplacian on $\mathbb{H}$, Theorem 3.2 implies that

$$(\pi_S(k)f_i)(z) = h(r_i)f_i(z) \tag{3.4}$$

for all $i \geq 0$ and for h and r_i as in Theorem 3.2. Hence the functions $f_0, f_1, f_2, \dots$ are a complete orthonormal system of eigenfunctions of the operator $\pi_S(k)$ with eigenvalues $h(r_i)$.

The next lemma shows that the operator $\pi_S(k)$ is trace-class. For a definition and important properties of trace-class operators we refer to the appendix.

Lemma 3.5. *The operator $\pi_S(k) : L^2_\mu(S) \to L^2_\mu(S)$ is a trace-class operator.*

Proof. Since the functions $f_0, f_1, f_2, \dots$ are a complete orthonormal system of eigenfunctions of the operator $\pi_S(k)$ with eigenvalues $h(r_i)$ as in the preceding paragraph, we conclude that the following convergence holds in L^2_μ:

$$K(z,w) \overset{L^2_\mu}{=} \sum_{i=0}^{\infty} h(r_i)f_i(z)f_i(w).$$

Applying the Laplacian to $z \mapsto K(z,w)$ yields

$$\triangle_z K(z,w) \overset{L^2_\mu}{=} \sum_{i=0}^{\infty} h(\lambda_j)r_i f_i(z)f_i(w).$$

As the function $\triangle_z K$ is continuous, the associated invariant integral operator is a Hilbert-Schmidt operator (see Proposition B.29) and hence

$$\sum_{i=0}^{\infty} |h(r_i)\lambda_i|^2 < \infty. \tag{3.5}$$

Since $\sum_{i=0}^{\infty} \lambda_i^{-2} < \infty$ we have with the Cauchy-Schwarz inequality

$$\begin{aligned}\left(\sum_{i=0}^{\infty} |h(r_i)|\right)^2 &= \left(\sum_{i=0}^{\infty} |h(r_i)\lambda_i\lambda_i^{-1}|\right)^2 \\ &\leq \sum_{i=0}^{\infty} |h(r_i)\lambda_i|^2 \sum_{i=0}^{\infty} \lambda_i^{-2} < \infty.\end{aligned}$$

So

$$\sum_{i=0}^{\infty} h(r_i)$$

is absolutely convergent. Thus by Proposition B.38 the operator $\pi_S(k)$ is indeed a trace-class operator. □

Using Proposition B.39 we hence derive the central relation

$$\sum_{i=0}^{\infty} h(r_i) = \operatorname{tr}(\pi_S(k)) = \int_D K(z,z)\, d\mu_{\mathbb{H}}(z) = \int_D \sum_{\gamma\in\Gamma} k(z,\gamma z)\, d\mu_{\mathbb{H}}(z).$$

As before, we will for convenience write $U(\cosh(\rho)) = k_{\mathbb{R}}(\rho)$. So we can write

$$\begin{aligned}\sum_{i=0}^{\infty} h(r_i) &= \sum_{\gamma\in\Gamma} \int_D k(z,\gamma z)\, d\mu_{\mathbb{H}}(z)\\ &= \sum_{\gamma\in\Gamma} \int_D U(\cosh(d(z,\gamma z))\, d\mu_{\mathbb{H}}(z)\\ &= \operatorname{area}(S)U(1) + \sum_{\gamma\neq \mathrm{id}} \int_D U(\cosh(d(z,\gamma z)))\, d\mu_{\mathbb{H}}(z), \qquad (3.6)\end{aligned}$$

where in the third line we separated the identity element from the sum taken over all elements of Γ. The Selberg trace formula is nothing more than a further transformation of equation (3.6).

Lemma 3.6. *In the above setting we have*

$$U(1) = \frac{1}{4\pi}\int_{-\infty}^{\infty} rh(r)\tanh(\pi r)\, dr.$$

Proof. By properties of the Abel transform we derive that

$$U(1+2u) = -\frac{1}{\pi}\int_u^{\infty} \frac{dq(v)}{\sqrt{v-u}}$$

with $q(v) = \frac{1}{2}g(2\log(\sqrt{v+1}+\sqrt{v}))$. Hence

$$\begin{aligned}U(1) &= -\frac{1}{\pi}\int_0^{\infty} \frac{q'(v)}{\sqrt{v}}\, dv\\ &= -\frac{1}{\pi}\int_0^{\infty} \frac{g'(2\log(\sqrt{v+1}+\sqrt{v}))}{\sqrt{v}}\frac{d}{dv}(2\log(\sqrt{v+1}+\sqrt{v}))\, dv\\ &= -\frac{1}{2\pi}\int_0^{\infty} \frac{g'(u)}{\sinh(u/2)}\, du\end{aligned}$$

where we used the substitution $u = 2\log(\sqrt{v+1}+\sqrt{v})$ in the last line. Since g and h are even, we get for almost all $u \in \mathbb{R}$

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iru}h(r)\, dr = \frac{1}{\pi}\int_0^{\infty} h(r)\cos(ru)\, dr.$$

As h lies in the Schwartz space,

$$g'(u) = -\frac{1}{\pi}\int_0^{\infty} rh(r)\sin(ru)\, dr.$$

So we conclude

$$U(1) = \frac{1}{2\pi^2}\int_0^{\infty}\int_0^{\infty} rh(r)\frac{\sin(ru)}{\sinh(u/2)}\, dudr.$$

Observe by the geometric series

$$\frac{1}{\sinh(u/2)} = \frac{2}{e^{u/2} - e^{-u/2}} = 2e^{-u/2} \sum_{n \geq 0} e^{-nu}$$

and so by two fold partial integration

$$\begin{aligned} \int_0^\infty \frac{\sin(ru)}{\sinh(u/2)}\, du &= 2 \sum_{n\geq 0} \int_0^\infty e^{-(2n+1)u/2} \sin(ru)\, du \\ &= 2 \sum_{n\geq 0} \frac{4r}{4r^2 + (2n+1)^2} \\ &= \sum_{n\in\mathbb{Z}} \frac{r}{r^2 + (n+1/2)^2}. \end{aligned}$$

Consider next the function $f(x) = e^{-2\pi r|x|}$. As we calculate in Example B.41, the Fourier transform of f is $\hat{f}(x) = \frac{1}{2\pi} \frac{2r}{r^2+x^2}$. We hence have

$$\frac{r}{r^2 + (n+1/2)^2} = \pi \hat{f}(n+1/2).$$

Denote by $F(x) := \hat{f}(x+1/2)$. Then, properties of the Fourier transform yield

$$\begin{aligned} \hat{F}(x) &= \int_{-\infty}^\infty \hat{f}(t+1/2) e^{2\pi i t x}\, dt \\ &= \int_{-\infty}^\infty \hat{f}(t) e^{2\pi i (t-1/2)x}\, dt \\ &= e^{\pi i x} f^{\wedge\wedge}(x) = e^{\pi i x} f(-x) \end{aligned}$$

So we derive, with the Poisson summation formula in the third line,

$$\begin{aligned} \int_0^\infty \frac{\sin(ru)}{\sinh(u/2)}\, du &= \sum_{n\in\mathbb{Z}} \frac{r}{r^2 + (n+1/2)^2} \\ &= \sum_{n\in\mathbb{Z}} \pi \hat{f}(n+1/2) \\ &= \sum_{n\in\mathbb{Z}} \pi e^{\pi i n} e^{-2\pi i |n|} \\ &= \pi \frac{1 - e^{-2\pi r}}{1 + e^{-2\pi r}} = \pi \tanh(\pi r). \end{aligned}$$

Thus

$$U(1) = \frac{1}{2\pi} \int_0^\infty r h(r) \tanh(\pi r)\, dt = \frac{1}{4\pi} \int_{-\infty}^\infty r h(r) \tanh(\pi r)\, dr.$$

$\square$

Lemma 3.7. *In the above setting,*

$$\sum_{\gamma\neq \mathrm{id}} \int_D U(\cosh(d(z,\gamma z)))\, d\mu_{\mathbb{H}}(z) = \sum_{\gamma\in G(S)} \frac{\ell(\delta)}{e^{\ell(\gamma)/2} - e^{-\ell(\gamma)/2}} g(\ell(\gamma)),$$

where $G(S)$ is the set of closed geodesics on S, $\ell(\gamma)$ is the displacement length of γ and δ is the corresponding prime element of γ such that $\gamma = \delta^n$ for $n \geq 1$ with displacement length $\ell(\delta)$. Furthermore, g is as before the Abel transform of $k(\operatorname{arcch}(t))$ multiplied by $\sqrt{2}$

$$g(u) = \sqrt{2} \int_x^\infty \frac{k_{\mathbb{R}}(\operatorname{arcch}(t))}{\sqrt{t-x}}\, dt.$$

Proof. Since conjugacy classes of elements of Γ correspond precisely to the set of closed geodesics, it suffices to consider the sum over some conjugacy class $C_\Gamma(\gamma^*)$ for $\gamma^* \in \Gamma$ not the identity. Hence the claim is implied by

$$\sum_{\gamma \in C_\Gamma(\gamma^*)} \int_D U(\cosh(d(z, \gamma z)))\, d\mu_{\mathbb{H}}(z) = \frac{\ell(\delta)}{e^{\frac{\ell(\gamma)}{2}} - e^{-\frac{\ell(\gamma)}{2}}} g(\ell(\gamma)).$$

Note that a term in the last sum looks like

$$\begin{aligned}\int_D U(\cosh\big(d(z, \gamma_1^{-1}\gamma^*\gamma_1 z)\big))\, d\mu_{\mathbb{H}}(z) &= \int_D U(\cosh(d(\gamma_1 z, \gamma^*\gamma_1 z)))\, d\mu_{\mathbb{H}}(z)\\ &= \int_{\gamma_1 D} U(\cosh(d(z, \gamma^* z)))\, d\mu_{\mathbb{H}}(z).\end{aligned}$$

Write

$$D_{\gamma^*} = \bigcup_{\gamma \in C_\Gamma(\gamma^*)} \gamma D,$$

and so

$$\sum_{\gamma \in C_\Gamma(\gamma^*)} \int_D U(\cosh(d(z, \gamma z)))\, d\mu_{\mathbb{H}}(z) = \int_{D_{\gamma^*}} U(\cosh(d(z, \gamma^* z)))\, d\mu_{\mathbb{H}}(z).$$

Observe that D_{γ^*} is a fundamental domain for the centralizer Γ_{γ^*}. By Proposition 1.35 we get that $\Gamma_{\gamma^*} = \{\delta^n : n \in \mathbb{Z}\}$ for some $\delta \in \Gamma$ such that $\gamma^* = \delta^m$ for some integer $m > 0$. Up to conjugation, we can assume that δ corresponds to an isometry given by $z \mapsto e^{\ell(\delta)} z$ for $\ell(\delta)$ the displacement length of δ. Note that the displacement length $\ell(\gamma)$ of γ is always given by $m\ell(\delta)$. Assume furthermore, after possibly replacing δ by its inverse, that $e^{\ell(\delta)} > 1$. Thus a fundamental domain for Γ_γ is given by the horizontal strip $1 < y < e^{\ell(\delta)}$. If we abbreviate

$2c := |e^{\frac{m\ell(\delta)}{2}} - e^{-\frac{m\ell(\delta)}{2}}|$, we conclude

$$\begin{aligned}
\int_{D_{\gamma^*}} U(\cosh(d(z,\gamma^* z)))\, d\mu_{\mathbb{H}}(z) &= \int_1^{e^{\ell(\delta)}} \int_{-\infty}^{\infty} U\Big(\cosh\Big(d(z, e^{m\ell(\delta)} z)\Big)\Big) \frac{dxdy}{y^2} \\
&= \int_1^{e^{\ell(\delta)}} \int_{-\infty}^{\infty} U(1 + 2(\tfrac{c|z|}{y})^2) \frac{dxdy}{y^2} \\
&= \left(\int_1^{e^{\ell(\delta)}} \frac{dy}{y} \right) \int_{-\infty}^{\infty} U(1 + 2c^2(x^2+1))\, dx \\
&= \frac{\ell(\delta)}{c} \int_{c^2}^{\infty} \frac{U(1+2u)}{\sqrt{u-c}}\, du \\
&= \frac{\ell(\delta)}{2c} g(m\ell(\delta)) \\
&= \frac{\ell(\delta)}{e^{\frac{\ell(\gamma)}{2}} - e^{-\frac{\ell(\gamma)}{2}}} g(\ell(\gamma)).
\end{aligned}$$

□

So Lemmas 3.6 and 3.7 plugged into equation (3.6) yield the Selberg trace formula

$$\sum_{i=0}^{\infty} h(r_i) = \frac{\text{Area}(S)}{4\pi} \int_{-\infty}^{\infty} r h(r) \tanh(\pi r)\, dr + \sum_{\gamma \in G(S)} \frac{\ell(\delta)}{e^{\frac{\ell(\gamma)}{2}} - e^{-\frac{\ell(\gamma)}{2}}} g(\ell(\gamma)).$$

Up to now, the functions h and g are given by a point pair invariant smooth function k of compact support. Since k is of compact support, so is g. In this formulation of the Selberg trace formula, note that the function k does not appear. Furthermore, the function g is a scalar multiple of the Abel transform of $k_{\mathbb{R}}$. Since the Abel transform is invertible, we can change the view point slightly by assuming the function g with compact support is given. Then we reconstruct the function k by the inverse of the Abel transform and h is simply given as the Fourier transform $\int_{-\infty}^{\infty} e^{iru} g(u)\, du$ of g.

The assumption that g is of compact support is quite strong. The next aim is to extend the Selberg trace formula to a wider class of functions than those of compact support. We call the functions we want to consider to be of *rapid decay*.

Definition 3.8. Let $\varepsilon > 0$ and consider

$$B_\varepsilon := \{z \in \mathbb{C} \,:\, |\text{Im}(z)| < \tfrac{1}{2} + \varepsilon\}$$

and assume that $h : B_\varepsilon \to \mathbb{C}$ is an even holomorphic function such that

$$h(r) = O(e^{\delta |\text{Re}(r)|^2}) \tag{3.7}$$

uniformly as $|r| \to \infty$ for some $\delta > 0$. Let g be the Fourier transform $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iru} h(u)\, du$ of h. Then the pair (h, g) is called of *rapid decay.*

Note that if g is of compact support, then the pair (h, g) is of rapid decay. Furthermore we note that for h a function satisfying the decay condition (3.7), the Abel transform is still invertible with the same inversion formula, which we leave as an exercise to check. We are now ready to state and prove a more general version of the Selberg trace formula.

Theorem 3.9. *(Selberg trace formula for functions of rapid decay) Let $S = \Gamma\backslash\mathbb{H}$ be a compact hyperbolic surface and denote by $G(S)$ the set of closed geodesics. Let (h, g) be a pair of rapid decay. Then*

$$\sum_{i=0}^{\infty} h(r_i) = \frac{\mathrm{Area}(S)}{4\pi} \int_{-\infty}^{\infty} r h(r) \tanh(\pi r)\, dr + \sum_{\gamma \in G(S)} \frac{\ell(\delta)}{e^{\frac{\ell(\gamma)}{2}} - e^{-\frac{\ell(\gamma)}{2}}} g(\ell(\gamma)),$$

where the series on both sides are absolutely convergent.

To prove the theorem we first need a statement about sequences that behave like the sequence of the ordered length of closed geodesics.

Lemma 3.10. *Let $0 < a_0 \leq a_1 \leq a_2 \leq \dots$ be a sequence that converges to infinity such that*

$$N_L(a) := \#\{n \,:\, 0 \leq a_n \leq L\} = O(e^L).$$

Then the sum

$$\sum_{n=0}^{\infty} \frac{a_n}{e^{(1+\varepsilon)a_n} - e^{\varepsilon a_n}}$$

converges for all $\varepsilon > 0$.

Proof. First note that for large enough a_n, say $a_n > L_1$, we have $e^{(1+\varepsilon)a_n} - e^{\varepsilon} \geq e^{(1+\varepsilon/2)a_n}$ and further for $\varepsilon > 0$ the function $\frac{a_n}{e^{(1+\varepsilon/2)a_n}}$ is decreasing, for $L > L_2$ large enough. Since a_n converges to infinity, there are only finitely many a_n such that $a_n \leq L_1$ and $a_n \leq L_2$. Denote by $L' = \max\{L_1, L_2\}$. It thus suffices to show that

$$\sum_{a_n \leq L'} \frac{a_n}{e^{(1+\varepsilon/2)a_n}}$$

converges. By assumption $N_L(a) \leq ce^L$ for some $c > 0$ and all $L > 0$. Since $\frac{a_n}{e^{(1+\varepsilon/2)a_n}}$ is decreasing for $L > L'$, we conclude

$$\begin{aligned}
\sum_{a_n \leq L'} \frac{a_n}{e^{(1+\varepsilon/2)a_n}} &= \sum_{L=L'}^{\infty} \sum_{L \leq a_n \leq L+1} \frac{a_n}{e^{(1+\varepsilon/2)a_n}} \\
&\leq \sum_{L=L'}^{\infty} \sum_{L \leq a_n \leq L+1} \frac{L}{e^{(1+\varepsilon/2)L}} \\
&= \sum_{L=L'}^{\infty} \frac{L}{e^{(1+\varepsilon/2)L}} (N_{L+1}(a) - N_L(a)) \\
&\leq \sum_{L=L'}^{\infty} \frac{L}{e^{(1+\varepsilon/2)L}} N_{L+1}(a) \\
&\leq \sum_{L=L'}^{\infty} \frac{L}{e^{(1+\varepsilon/2)L}} ce^{L+1} \\
&= ce \sum_{L=L'}^{\infty} \frac{L}{e^{\varepsilon L/2}} < \infty.
\end{aligned}$$

□

We are now ready to prove the Selberg trace formula for functions of rapid decay.

Proof. (of the Selberg trace formula for functions of rapid decay) We fist show that both sides of the series are absolutely convergent. By assumption,

$$|h(r)| \leq ce^{-\delta|\mathrm{Re}(r)|^2}$$

for some constant $c > 0$. Since $\sum_{i=0}^{\infty} \lambda_i^{-2} < \infty$ we conclude

$$\sum_{i=0}^{\infty} |h(r_i)| < \sum_{i=0}^{\infty} ce^{-\delta|\mathrm{Re}(r_i)|^2} < \infty$$

and hence the left-hand side is absolutely convergent.

Let h be defined on B_ε for $\varepsilon > 0$. By assumption g is given by the Fourier transform $g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iru}h(r)\,dr$. Let $\varepsilon' \in (0, \varepsilon)$ and assume without loss of generality $u < 0$. Then the Cauchy integral theorem yields for $E' = \mathbb{R} + i(\frac{1}{2} + \varepsilon')$ that

$$g(u) = \frac{1}{2\pi}\int_{E'} e^{-iru}h(r)\,dr$$

and hence

$$\begin{aligned} |g(u)| &\leq \frac{ce^{-(1/2+\varepsilon')|u|}}{2\pi}\int_{-\infty}^{\infty} e^{-\delta|\mathrm{Re}(r)|^2}\,dr \\ &\leq \frac{ce^{-(1/2+\varepsilon')|u|}}{2\pi}. \end{aligned}$$

Hence for the absolute values of the series on the right-hand side we have

$$\begin{aligned} \sum_{\gamma\in G(S)} \frac{\ell(\delta)}{e^{\frac{\ell(\gamma)}{2}} - e^{-\frac{\ell(\gamma)}{2}}}|g(\ell(\gamma))| &\leq \frac{c}{2\pi}\sum_{\gamma\in G(S)} \frac{\ell(\delta)}{e^{\frac{\ell(\gamma)}{2}} - e^{-\frac{\ell(\gamma)}{2}}}e^{-(1/2+\varepsilon')\ell(\gamma)} \\ &= \frac{c}{2\pi}\sum_{\gamma\in G(S)} \frac{\ell(\delta)}{e^{(1+\varepsilon')\ell(\gamma)} - e^{\varepsilon'\ell(\gamma)}}. \end{aligned}$$

In Lemma 1.34 we proved $c_L(S) = O(e^L)$. Hence we can apply Lemma 3.10 to conclude that the last sum is finite. So the sum on the right-hand side is also absolutely convergent.

It remains to prove that the left-hand and right-hand side are equal. To simplify the notation we abbreviate:

$$\mathscr{L}(h) := -\frac{\mathrm{Area}(S)}{4\pi}\int_{-\infty}^{\infty} rh(r)\tanh(\pi r)\,dr + \sum_{i=0}^{\infty} h(r_i) \tag{3.8}$$

$$\mathscr{R}(h) := \sum_{\gamma\in G(S)} \frac{\ell(\delta)}{e^{\frac{\ell(\gamma)}{2}} - e^{-\frac{\ell(\gamma)}{2}}}g(\ell(\gamma)) \tag{3.9}$$

The aim is to prove $\mathscr{L}(h) = \mathscr{R}(g)$. We already proved this in the case where g is of compact support. We proceed by approximating g by functions of compact support.

For this let $\varphi : [0, \infty) \to \mathbb{R}$ be a smooth function such that

$$\varphi(x) = \begin{cases} 1 & \text{for } x \in [0,1], \\ \text{monotonly decreasing} & \text{for } x \in [1,2], \\ 0 & \text{for } x \in [2,\infty). \end{cases}$$

For $m \in \mathbb{N}$ we define

$$\varphi_m(u) := \begin{cases} 1 & \text{for } |u| \le m, \\ \varphi(|u| - m) & \text{for } |u| \ge m. \end{cases}$$

We further denote for $u \in \mathbb{R}$ and $r \in \mathbb{C}$,

$$g_m(u) = g(u)\varphi_m(u), \qquad h_m(r) = \int_{-\infty}^{\infty} e^{iru} g_m(u)\, du.$$

It follows that g_m is an even function satisfying $g_m = g$ on $[0, m]$. We hence have that

$$\begin{aligned} |\mathscr{R}(g) - \mathscr{R}(g_m)| \le & \sum_{m \le \ell(\gamma)} \frac{\ell(\delta)}{e^{\frac{\ell(\gamma)}{2}} - e^{-\frac{\ell(\gamma)}{2}}} |g(\ell(\gamma))| \\ & + \sum_{m \le \ell(\gamma)} \frac{\ell(\delta)}{e^{\frac{\ell(\gamma)}{2}} - e^{-\frac{\ell(\gamma)}{2}}} |g_m(\ell(\gamma))| \end{aligned}$$

Since $|g_m| \le |g|$ and $\mathscr{R}(g)$ converges absolutely, we conclude $\mathscr{R}(g_m) \to \mathscr{R}(g)$ as $g \to \infty$.

Now we prove $\mathscr{L}(h_m) \to \mathscr{L}(h)$ as $m \to \infty$. For the first part of $\mathscr{L}$ this follows from dominated convergence since $|h_m| \le |h|$ for all m. Hence it suffices to show that

$$\sum_{i=0}^{\infty} h_m(r_n) \to \sum_{i=0}^{\infty} h(r_i) \qquad \text{as } m \to \infty. \tag{3.10}$$

To see this, note that for the fourth derivative we also have

$$g_m^{(4)}(u) = O(e^{-(1/2+\varepsilon)|u|}),$$

uniformly in m and r. Thus, by integrating by parts four times we get

$$h_m(r) = O((1 + |r|)^{-4})$$

uniformly in m and r. We further have

$$\sum_{i=1}^{\infty} \frac{1}{(1 + |r|)^4} \le \sum_{i=1}^{\infty} \frac{1}{(|r|)^4} < \infty,$$

because $\sum_{i=1}^{\infty} \lambda^{-2} < \infty$. By pointwise convergence $h_m(r) \to h(r)$ for $m \to \infty$ for all r, we conclude by dominated convergence $\mathscr{L}(h_m) \to \mathscr{L}(h)$. By the case of compact support we conclude $\mathscr{L}(h) = \mathscr{R}(g)$. $\square$

Remark. One can even prove the Selberg trace formula for pairs (h, g) where h satisfies the decay property

$$h(r) = O(1 + |r|^2)^{-1-\varepsilon}.$$

Assuming the result of the next section, we can prove the Selberg trace formula for such pairs (h, g) in a very similar way to the proof given above for functions of rapid decay. More precisely, one can approximate such functions (g, h) with functions of rapid decay. We refrain from making this more explicit, since for our purposes the case of rapid decay is sufficient. To see a complete proof we refer to [Bus92].

3.3 The Weyl Law

As a first application of the Selberg trace formula we prove the Weyl law.

Theorem 3.11. *(The Weyl law) Let $S = \Gamma\backslash\mathbb{H}$ be a compact hyperbolic surface with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$, then*

$$\#\{n : 0 \leq \lambda_n \leq T\} \sim \frac{\operatorname{area}(S)}{4\pi} T.$$

We first prove a lemma concerning sequences with some special properties.

Lemma 3.12. *Let $(a_n)_{n=0}^\infty$ be a sequence of positive numbers that converges to infinity such that*

$$\sum_{n=0}^{\infty} e^{-ra_n} \sim \frac{c}{r}$$

as $r \to 0$. Then

$$\#\{n : 0 \leq a_n \leq T\} \sim c \cdot T,$$

as $T \to \infty$.

Proof. For $r > 0$ consider the measure

$$\mu_r := \sum_{n=0}^{\infty} r\delta_{ra_n}.$$

Note that this is a positive Radon measure on $[0, \infty)$ since a_n converges to infinity. Denote

$$D := \sup_{0<r\leq 1} r \sum_{n=0}^{\infty} e^{-ra_n}.$$

We claim that for all $T > 1$ and all $r > 0$, we have $\mu_r([0, T]) \leq e \cdot D \cdot T$. To prove the claim, observe that if $a_n \leq \frac{1}{r}$ then equivalently $-1 \leq -ra_n$ and thus $e^{-1} \leq e^{-ra_n}$. Hence for $r \in (0, 1]$

$$r \cdot \#\{n : a_n \leq \tfrac{1}{r}\} \cdot e^{-1} \leq r \sum_{n=0}^{\infty} e^{-ra_n} \leq D.$$

So we conclude

$$\mu_r([0,T]) = r \cdot \#\{n : a_n \leq \tfrac{T}{r}\} \leq e \cdot D \cdot T.$$

Hence $\{\mu_r : 0 < r \leq 1\}$ is compact in the weak*-topology. Thus we can choose suitable $r_n \to 0$ such that the measures μ_{r_n} converge in the weak*-topology to a measure μ as $n \to \infty$. Moreover, for all $x > 0$ we have

$$\int_0^\infty e^{-tx}\, d\mu_{r_n}(t) = r_n \sum_{n=0}^\infty e^{-xr_n a_n}.$$

So, if $r_n \to 0$, we have

$$r_n \sum_{n=0}^\infty e^{-xr_n a_n} \sim r_n \cdot \frac{c}{xr_n} = \frac{c}{x}.$$

Since the right hand side does not depend on r_n, we conclude that the right hand side converges to the left hand side as $r_n \to 0$. Thus, if $r_n \to 0$,

$$\int_0^\infty e^{-tx}\, d\mu(t) = \frac{c}{x}.$$

Denote next by

$$\hat{\mu}(s) := \int_0^\infty e^{-st}\, d\mu(t)$$

the Fourier transform of the measure μ. This is a holomorphic function in $\mathrm{Re}(s) > 0$ and as shown above $\hat{\mu}(s) = \frac{c}{s}$ on $(0,\infty)$. Thus $\hat{\mu}(s) = \frac{c}{s}$ for all $\mathrm{Re}(s) > 0$. Therefore $\mu = c \cdot dt$, where dt is the Lebesgue measure. Hence

$$\lim_{n\to\infty} \mu_{r_n} = c \cdot dt$$

in the weak*-topology. □

Proof. (of Theorem 3.11) We consider the admissible pair $h(r) = e^{-\delta r^2}$ and $g(x) = (4\pi\delta)^{-1/2} e^{-\frac{x^2}{4\delta}}$, where δ is a positive parameter. Then by the Selberg trace formula,

$$\begin{aligned}\sum_{i=0}^\infty e^{-\delta r_i^2} = {} & \frac{\mathrm{Area}(S)}{4\pi} \int_{-\infty}^\infty re^{-\delta r^2} \tanh(\pi r)\, dr \\ & + \frac{1}{\sqrt{4\pi\delta}} \sum_{\gamma \in G(S)} \frac{\ell(\delta)}{e^{\frac{\ell(\gamma)}{2}} - e^{-\frac{\ell(\gamma)}{2}}} e^{-\frac{\ell(\gamma)^2}{4\delta}}. \end{aligned} \tag{3.11}$$

We will first estimate the second term on the right-hand side of (3.11) in order to show that it goes to zero as $\delta \to 0$. Denote by ℓ_0 the length of the shortest closed geodesic on S. First note that if $\delta < \frac{\ell_0}{8}$, then

$$\frac{\ell(\gamma)^2}{4\delta} \geq \frac{\ell(\gamma)^2}{8\delta} + \ell(\gamma) \geq \frac{\ell_0}{8\delta} + \ell(\gamma).$$

Hence

$$\frac{1}{\sqrt{4\pi\delta}} \sum_{\gamma \in G(S)} \frac{\ell(\delta)}{e^{\frac{\ell(\gamma)}{2}} - e^{-\frac{\ell(\gamma)}{2}}} e^{-\frac{\ell(\gamma)^2}{4\delta}} \leq \frac{1}{\sqrt{4\pi\delta}} e^{-\frac{\ell_0}{4\delta}} \sum_{\gamma \in G(S)} \frac{\ell(\delta)}{e^{\frac{\ell(\gamma)}{2}} - e^{-\frac{\ell(\gamma)}{2}}} e^{-\ell(\gamma)}. \tag{3.12}$$

We show that the sum

$$\sum_{\gamma\in G(S)} \frac{\ell(\delta)}{e^{\frac{\ell(\gamma)}{2}} - e^{-\frac{\ell(\gamma)}{2}}} e^{-\ell(\gamma)} \tag{3.13}$$

is finite. Observe

$$\sum_{\gamma\in G(S)} \frac{\ell(\delta)}{e^{\frac{\ell(\gamma)}{2}} - e^{-\frac{\ell(\gamma)}{2}}} e^{-\ell(\gamma)} \leq \sum_{\gamma\in G(S)} \frac{\ell(\gamma)}{e^{\frac{\ell(\gamma)}{2}} - e^{-\frac{\ell(\gamma)}{2}}} e^{-\ell(\gamma)}.$$

By Lemma 1.34 we have $c_S(L) = O(e^L)$. Hence by Lemma 3.10 for $\varepsilon = \frac{1}{2}$ we conclude that the sum (3.13) is finite. As $\delta \to 0$ the second term of the right hand side of (3.11) converges to zero by (3.12).

Integrating by parts gives

$$\int_{-\infty}^{\infty} re^{-\delta r^2} \tanh(\pi r)\, dr = \frac{1}{\delta} \int_{-\infty}^{\infty} e^{-\delta r^2} \frac{\pi}{2\cosh(\pi r)^2}.$$

By considering the Taylor expansion of the exponential function, we get

$$\begin{aligned}
\int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(-\delta r^2)^n}{n!} \frac{\pi}{2\cosh(\pi r)^2} &= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{(-\delta r^2)^n}{n!} \frac{\pi}{2\cosh(\pi r)^2} \\
&= \int_{-\infty}^{\infty} \frac{\pi}{2\cosh(\pi r)^2}\, dr \\
&+ \sum_{i=1}^{\infty} \int_{-\infty}^{\infty} \frac{(-\delta r^2)^n}{n!} \frac{\pi}{2\cosh(\pi r)^2}\, dr \\
&= 1 - \delta \int_{-\infty}^{\infty} \sum_{i=1}^{\infty} \frac{(-\delta)^{n-1}}{n!} r^{2n} \frac{\pi}{2\cosh(\pi r)^2}\, dr.
\end{aligned}$$

We take a closer look at the second term in the last line. We have

$$\begin{aligned}
0 &\leq \int_{-\infty}^{\infty} \sum_{i=1}^{\infty} \frac{(-\delta)^{n-1}}{n!} r^{2n} \frac{\pi}{2\cosh(\pi r)^2}\, dt \\
&\leq \int_{-\infty}^{\infty} r^2 \sum_{i=1}^{\infty} \frac{(-\delta)^{n-1}}{(n-1)!} r^{2(n-1)} \frac{\pi}{2\cosh(\pi r)^2}\, dr \\
&= \int_{-\infty}^{\infty} r^2 e^{-\delta r^2} \frac{\pi}{2\cosh(\pi r)^2}\, dr \\
&\leq 1
\end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} e^{-\delta r^2} \frac{\pi}{2\cosh(\pi r)^2}\, dr = 1 + O(\delta)$$

Thus one obtains

$$\int_{-\infty}^{\infty} re^{-\delta r^2} \tanh(\pi r)\, dr = \frac{1}{\delta}(1 + O(1))$$

So we proved
$$\sum_{j=0}^{\infty} e^{-\delta r_i^2} = \frac{\text{area}(S)}{4\pi\delta} + \frac{\text{area}(S)}{4\pi} O(1)$$
as $\delta \to 0$ and hence
$$\sum_{j=0}^{\infty} e^{-\delta r_i^2} \sim \frac{\text{area}(S)}{4\pi\delta}$$
as $\delta \to 0$. Thus the Weyl law follows from Lemma 3.10. □

3.4 The Prime Geodesic Theorem

We follow mostly [Ber11]. As in the previous sections, let $S = \Gamma\backslash\mathbb{H}$ a compact hyperbolic surface. Denote further by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ the eigenvalues of the Laplacian. For $\lambda_j \leq \frac{1}{4}$ we write

$$s_j := \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_j}.$$

As in section 1.2 write $c_S(L)$ for the number of closed geodesics of length less that L. In analogy to the prime number counting function, we will also be interested in the function

$$\pi_S(L) := \#\{\text{prime geodesics on } S \text{ with length less than} \log(L)\}.$$

We also use li for the logarithmic integral

$$\text{li}(L) = \int_2^L \frac{dt}{\log(t)}.$$

We are now finally able to state and prove the prime geodesic theorem.

Theorem 3.13. *(Prime geodesic theorem) As L tends to infinity,*

$$\pi_S(L) = \text{li}(L) + \sum_{3/4 < s_j < 1} \text{li}(L^{s_j}) + O\left(\frac{L^{3/4}}{\log(L)}\right).$$

To prove the prime geodesic theorem, we consider the following function

$$\begin{aligned} H(T) &:= \sum_{\ell(\gamma) \leq T} \frac{\ell(\delta)}{e^{\ell(\gamma)/2} - e^{\ell(\gamma)/2}} 2\cosh\left(\frac{1}{2}\ell(\gamma)\right) \\ &= \sum_{\ell(\gamma)\leq T} \ell(\delta)(1 + e^{-\ell(\gamma)})(1 - e^{-\ell(\gamma)})^{-1} \end{aligned}$$

where we again denote by δ the unique primitive closed geodesic associated to γ. Further, denote
$$E_T(\alpha) := \frac{e^{T\alpha}}{\alpha}.$$
We first study the asymptotic behavior of $H(T)$ with the help of the Selberg trace formula and then prove the prime geodesic theorem.

Lemma 3.14. *As T tends to infinity,*

$$H(T) = E_T(1) + \sum_{3/4<s_j<1} E_T(s_j) + O(e^{3T/4}).$$

Proof. Denote by $\chi_{[-T,T]}$ the characteristic function of $[-T,T]$ in $\mathbb{R}$ and by

$$g_T(x) = 2\cosh(x/2)\chi_{[-T,T]}(x).$$

Furthermore denote by φ an even nonnegative smooth function with support contained in $[-1,1]$ and such that $\int_{-\infty}^{\infty}\varphi(x) = 1$. Given a real number $\varepsilon > 0$, write

$$\varphi_\varepsilon(x) := \frac{\varphi(x/\varepsilon)}{\varepsilon}.$$

Hence φ_ε is supported in $[-\varepsilon,\varepsilon]$ and has total mass 1. Finally define

$$g_T^\varepsilon = (g_T * \varphi_\varepsilon)(x) = 2\int_{-\infty}^{\infty}\cosh((x-y)/2)\chi_{[-T,T]}(x-y)\varphi_\varepsilon(y)\,dy.$$

For any $\varepsilon, T > 0$ the function g_T^ε is smooth and has compact support. We denote by h_T^ε the Fourier transform of g_T^ε.

In this part of the thesis we use the convention that $\hat{f}(r) = \int_{-\infty}^{\infty} e^{-irx}f(x)\,dx$ and we have $\widehat{f_1 * f_2} = \hat{f}_1\hat{f}_2$ and $2\pi f^{\wedge\wedge} = f$ if f is even. Observe, for $S(w) := 2w^{-1}\sinh(Tw)$ with the convention $S(0) = 2T$,

$$\begin{aligned}\hat{g}_T(r) &= 2\int_{-T}^{T} e^{-irx}\cosh(x/2)\,dx\\ &= \int_{-T}^{T} e^{(1/2-ir)x} + e^{-(1/2+ir)x}dx\\ &= S(1/2+ir) + S(1/2-ir).\end{aligned}$$

Thus

$$h_T^\varepsilon = (S(1/2+ir) + S(1/2-ir))\hat{\varphi}_\varepsilon(r).$$

We next define the function H_ε as follows:

$$H_\varepsilon(T) := \sum_{\gamma\in G(S)} \frac{\ell(\delta)}{e^{\ell(\gamma)/2} - e^{\ell(\gamma)/2}} g_T^\varepsilon(\ell(\gamma)).$$

Note that since the support of φ_ε is contained in $[-\varepsilon,\varepsilon]$ we only need to take the sum over all the geodesics with length $\ell(\gamma) \le T+\varepsilon$. Thus

$$H_\varepsilon(T) := \sum_{\ell(\gamma)\le T+\varepsilon} \frac{\ell(\delta)}{e^{\ell(\gamma)/2} - e^{\ell(\gamma)/2}} g_T^\varepsilon(\ell(\gamma)). \tag{3.14}$$

The Selberg trace formula implies

$$\begin{aligned}H_\varepsilon(T) &= \sum_{i=0}^{\infty} h_T^\varepsilon(r_i) - \frac{\text{area}(S)}{4\pi}\int_{-\infty}^{\infty} r h_T^\varepsilon(r)\tanh(\pi r)\,dr\\ &= \sum_{\lambda_i<\frac14} h_T^\varepsilon(r_i) + \sum_{\lambda_i\ge\frac14} h_T^\varepsilon(r_i) - \frac{\text{area}(S)}{4\pi}\int_{-\infty}^{\infty} r h_T^\varepsilon(r)\tanh(\pi r)\,dr.\end{aligned}$$

For the first part of this sum, note that the values r_j are purely imaginary since $r_i^2 + 1/4 = \lambda_i$. So if $\lambda_i < \frac{1}{4}$, we have $r_i = i\sqrt{\frac{1}{4} - \lambda_i}$ and hence fo $s_j = \frac{1}{2} - ir_i$ we have $s_i \in (\frac{1}{2}, 1]$. Furthermore, observe by inserting the Taylor expansion of e^x that $\hat{\varphi}_\varepsilon(x) = \hat{\varphi}(\varepsilon x) = 1 + O(\varepsilon x)$ as $\varepsilon \to 0$. Hence for $\lambda_i < \frac{1}{4}$

$$\begin{aligned} h_T^\varepsilon(r_i) &= (S(s_i + 2ir_i) + S(s_i))\hat{\varphi}_\varepsilon(r_i) \\ &= (S(s_i - 2\sqrt{\tfrac{1}{4} - \lambda_i}) + S(s_i))(1 + O(\varepsilon r_i)) \\ &= E_T(s_i) + O(\varepsilon e^{s_j T}). \end{aligned}$$

It will later turn out to be useful to only consider the terms $3/4 < s_j < 1$. We conclude that

$$\sum_{\lambda_i < \frac{1}{4}} h_T^\varepsilon(r_i) = E_T(1) + \sum_{3/4 < s_j < 1} (E_T(s_j) + O(\varepsilon e^{s_j T})).$$

We next write

$$\sum_{\lambda_i \geq 1/4} h_T^\varepsilon(r_i) - \frac{\text{area}(S)}{4\pi} \int_{-\infty}^{\infty} r h_T^\varepsilon(r) \tanh(\pi r)\, dr = \int_0^\infty h_T^\varepsilon(r) dm(r)$$

for the measure $dm(r) = \sum_{r_i > 0} \delta_{r_i} - \frac{\text{area}(S)}{2\pi} r \tanh(\pi r) dr$, where δ_{r_i} is the Dirac-measure at r_i. Since $\hat{\varphi}(\rho) = O((1 + |\rho|^{-2}))$ and $|\hat{g}_T(r)| = O((1 + r)^{-1} e^{T/2})$ we conclude

$$|h_T^\varepsilon(r)| \leq c e^{T/2} (1 + r)^{-1} (1 + \varepsilon r)^{-2}$$

for $c > 0$ a constant. Thus

$$\left| \int_0^\infty h_T^\varepsilon(t)\, dm(r) \right| \leq c \cdot e^{T/2} \int_0^\infty (1 + r)^{-1} (1 + \varepsilon r)^{-2}\, |dm(r)|.$$

To bound this last integral, we split up $\int_0^\infty = \int_0^{1/\varepsilon} + \int_{1/\varepsilon}^\infty$ and integrate by parts. By the Weyl law $\mu(r) = O(r^2)$, hence the integral is bounded by $O(\varepsilon^{-1} e^{T/2})$.

Note that since

$$g_{T-\varepsilon}^\varepsilon(x) \leq 2\cosh(x/2)\chi_{[-T,T]} \leq g_{T+\varepsilon}^\varepsilon(x) \tag{3.15}$$

for all $x > 0$, we have

$$H_\varepsilon(T - \varepsilon) \leq H(T) \leq H_\varepsilon(T + \varepsilon) \tag{3.16}$$

for all T and $\varepsilon > 0$.

We consider now the particular case $\varepsilon = e^{-T/4}$. Then

$$H_\varepsilon(T) = E_T(1) + \sum_{3/4 < s_j < 1} E_T(s_j) + O(e^{3T/4}). \tag{3.17}$$

Furthermore for a fixed $\alpha \in (3/4, 1]$ we have

$$E_{T \pm \varepsilon}(\alpha) = \alpha^{-1} e^{(T \pm \varepsilon)\alpha} = \alpha^{-1} e^{T\alpha}(1 + O(\varepsilon)) = E_T(\alpha) + O(e^{3T/4}).$$

By combining (3.16) and (3.17) we derive our claim

$$H(T) = E_T(1) + \sum_{3/4 < s_j < 1} E_T(s_j) + O(e^{3T/4}).$$

□

In addition, we will need the following statement concerning sequences.

Lemma 3.15. *Let $0 < a_0 \leq a_1 \leq a_2 \leq \dots$ be a sequence that converges to infinity. Then*

$$\lim_{n\to\infty} \frac{\sum_{i=0}^n a_i e^{-a_i}}{\sum_{i=0}^n a_i} = 0.$$

Proof. We first claim that

$$\lim_{n\to\infty} \frac{\sum_{i=0}^n a_i^{-1}}{\sum_{i=0}^n a_i} = 0.$$

To prove the claim let $\varepsilon > 0$. Then choose $n_0 \geq 1$ such that $\frac{1}{n_0} \leq \varepsilon$. Hence

$$\begin{aligned} \lim_{n\to\infty} \frac{\sum_{i=0}^n a_i^{-1}}{\sum_{i=0}^n a_i} &= \lim_{n\to\infty} \frac{\sum_{a_n \leq n_0} a_i^{-1}}{\sum_{i=0}^n a_i} + \frac{\sum_{a_n \geq n_0} a_i^{-1}}{\sum_{i=0}^n a_i} \\ &\leq \lim_{n\to\infty} \frac{\sum_{a_n \leq n_0} a_i^{-1}}{\sum_{i=0}^n a_i} + \frac{\sum_{a_n \geq n_0} \varepsilon}{\sum_{i=0}^n a_i} \leq \varepsilon \end{aligned}$$

since $a_n \to \infty$. The statement follows since for a_n large enough, we have that $a_n^2 \leq e^{a_n}$. □

Combining all this we can prove the prime geodesic theorem.

Proof. (of Theorem 3.13) Observe that

$$\begin{aligned} H(T) &= \sum_{\ell(\gamma)\leq T} \ell(\delta)(1 + e^{-\ell(\gamma)})(1 + e^{-\ell(\gamma)}) \\ &= \sum_{\ell(\gamma)\leq T} \ell(\delta)(1 + 2e^{-\ell(\gamma)} + e^{-2\ell(\gamma)}) \\ &= \sum_{\ell(\gamma)\leq T} \ell(\delta) + O\left(\sum_{\ell(\gamma)\leq T} \ell(\delta) e^{-\ell(\gamma)} \right). \end{aligned}$$

Next, consider the function

$$\psi(T) := \sum_{\ell(\gamma)\leq T} \ell(\delta).$$

By Lemma 3.14 there is no upper bound on $\ell(\delta)$ since otherwise the growth rate would be polynomial. So we can view $\ell(\delta)$ as a sequence tending to infinity. Thus by Lemma 3.15 we have that

$$H(T) = \psi(T) + o(\psi(T)).$$

Hence $H(T)$ and $\psi(T)$ have the same asymptotic expansion and so we deduce

$$\psi(T) = E_T(1) + \sum_{3/4 < s_j < 1} E_T(s_j) + O(e^{3T/4}).$$

We next investigate the function

$$\theta(T) = \sum_{\substack{\ell(\gamma)\leq T \\ \gamma \text{ prime}}} \ell(\gamma).$$

Denote by ℓ_0 the length of the shortest closed geodesic on S. As soon as $k > T/\ell_0$, we have

$$\psi(T) = \theta(T) + \theta(T/2) + \ldots + \theta(T/k).$$

If we denote by

$$m(T) = \operatorname{int}\left(T/\ell_0\right),$$

the integer part of $\frac{T}{\ell_0}$, we can also write

$$\psi(T) = \theta(T) + \sum_{m=2}^{m(T)} \theta\left(T/m\right)$$

and so we can bound

$$\psi(T) \leq \theta(T) + m(T)\theta\left(T/2\right).$$

Note further

$$m(T)\theta(T/2) \leq \frac{t}{\ell_0}\psi(T/2) = O(te^{T/2}).$$

Thus we derive that

$$\theta(T) = E_T(1) + \sum_{3/4<s_j<1} E_T(s_j) + O(e^{3T/4}).$$

Since $\pi_S(L) = \int_\delta^{\log(L)} T^{-1}\, d\theta(T)$ if $\delta < \ell_0$, we conclude

$$\pi_S(L) = \operatorname{li}(L) + \sum_{3/4<s_j<1} \operatorname{li}(L^{s_j}) + O\left(\frac{L^{3/4}}{\log(L)}\right).$$

$\square$

Corollary 3.16. *For the number of closed geodesics of length less than L,*

$$c_S(L) \sim \frac{e^L}{L}$$

as L tends to infinity.

Proof. We show the equivalent statement

$$c_S(\log(L)) \sim \frac{L}{\log(L)},$$

as L tends to infinity. As before, for $k > T/\ell_0$ where ℓ_0 is the length of the shortest closed geodesic on S, we have

$$c_S(\log(L)) = \pi_S(L) + \pi_S(L/2) + \ldots + \pi_S(L/k)$$

and hence $c_S(\log(L))$ and $\pi_S(L)$ have the same asymptotic expansion. The statement now follows since

$$\operatorname{li}(L) \sim \frac{L}{\log(L)},$$

as $L \to \infty$. $\square$

A Topological Groups

A.1 Topological Groups, Haar Measures and Discrete Subgroups

Definition A.1. A group G endowed with a topology is called a *topological group* if the product map $G \times G \to G$ is continuous with respect to the product topology and the inverse map $G \to G$ is continuous too.

Definition A.2. Let G be a topological group and let X be a topological space. A *continuous group action* of G on X is a continuous map

$$\cdot : G \times X \to X, \qquad (g, x) \mapsto g \circ x$$

with $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in X$ and $e \circ x = x$ for all $x \in X$, where $e \in G$ is the identity element.

Definition A.3. Let G be a topological group. A *left Haar measure* is a Borel measure μ_G that assigns a finite number to every compact set and a number bigger than zero to every open set and that satisfies for every measurable function $f : G \to \mathbb{C}$ and $g \in G$,

$$\int_G f(gx)\, d\mu_G(x) = \int_G f(x)\, d\mu_G(x).$$

We call a Borel measure μ a *right Haar measure* if it satisfies the first two properties of a left Haar measure, but satisfies for every measurable function $f : G \to \mathbb{C}$ and $g \in G$,

$$\int_G f(xg)\, d\mu_G(x) = \int_G f(x)\, d\mu_G(x).$$

Definition A.4. We call a topological group G *unimodular* if every left Haar measure is also a right Haar measure.

Theorem A.5. *(See [IZ17] for a discussion or Chapter 10 of [EW17]) On every locally compact Hausdorff group, there exists a left (or right) Haar measure which is unique up to scalar multiples.*

Definition A.6. A topological group G is called *discrete* if the topology on G is the discrete topology, i.e. every set is open and closed.

Definition A.7. A subgroup H of a topolocial group G is called *discrete* if the induced topology on H is the discrete topology.

Lemma A.8. *A subgroup Γ of a locally compact topological group G is discrete if and only if there does not exist a sequence of elements $\gamma_n \in \Gamma$ with $\gamma_n \neq e$ such that $\gamma_n \to e$ as $n \to \infty$.*

Proof. Assume that Γ is a discrete subgroup. Then, if such a sequence converges to e, for all n large enough $\gamma_n = e$, contradicting the assumption. Contrarily assume that Γ is not discrete. Then such a sequence exists. □

A.2 Lie Groups and Riemannian Symmetric Pairs

Definition A.9. A topological group G with the structure of a smooth manifold such that multiplication $G \times G \to G$ and inversion $G \to G$ are smooth is called a *Lie group.*

Definition A.10. Let G be a Lie group with identity $e \in G$. The tangent space at the identity

$$\mathfrak{g} := T_e G$$

is called the *Lie algebra* of G.

Definition A.11. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Denote by $\mathrm{int}(g) : G \to G, x \mapsto gxg^{-1}$ and note that $\mathrm{int}(g)$ is a smooth automorphism. The map

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$$

given by

$$\mathrm{Ad}(g) := D_e \mathrm{int}(g) : \mathfrak{g} \to \mathfrak{g},$$

where D_e denotes the derivative at e, is called the *adjoint representation.*

Proposition A.12. *The adjoint representation is analytic.*

Proof. See [Hel01] page 127. □

Definition A.13. Let G be a connected Lie group and let $K \leq G$ be a closed subgroup of G. Then (G, K) is called a *Riemannian symmetric pair* if

(i) $\mathrm{Ad}_G(K) \leq \mathrm{GL}(\mathfrak{g})$ is compact, and

(ii) there exists an involution $\sigma : G \to G$ with $(G^\sigma)^\circ \subset K \subset G^\sigma$, where an involution satisfies $\sigma^2 = \mathrm{id}$ and we denote by

$$G^\sigma := \{g \in G \,:\, m\sigma(g) = g\}$$

and by $(G^\sigma)^\circ$ the connected component of G^σ at the identity.

Example A.14. Let $G = \mathrm{SL}_2(\mathbb{R})$ and $K = \mathrm{SO}_2(\mathbb{R})$ and $\sigma : G \to G, g \mapsto (g^{-1})^T$. We then have

$$G^\sigma = \{g \in \mathrm{SL}_2(\mathbb{R}) \,:\, g^T g = \mathrm{Id}\} = \mathrm{SO}_2(\mathbb{R}) = K.$$

Since K is compact and the adjoint representation is analytic and hence continuous, we conclude that $\mathrm{Ad}_G(K) \leq \mathrm{GL}(\mathfrak{g})$ is compact. Hence (G, K) is a Riemannian symmetric pair.

B Functional Analysis

Everything in this appendix can be found either in the book on Functional Analysis by Einsiedler and Ward [EW17] or in the one of Zimmer [Zim90]. We refrain from giving proofs, except in a few selected cases.

B.1 Banach Spaces

Definition B.1. A normed vector space $(V, ||\cdot||)$ is a *Banach space* if V is complete with respect to the norm $||\cdot||$.

Definition B.2. A Banach space $(V, ||\cdot||)$ is called *separable* if there is a countable dense subset.

Theorem B.3. *(Stone-Weierstrass, Theorem 2.40 of [EW17],) Let X be a compact metric space and $\mathscr{A} \subset C(X)$ a linear subspace of the space of continuous functions from X to $\mathbb{C}$. Suppose further that*

(i) $\mathscr{A}$ is a subalgebra i.e. closed under multiplication.

(ii) $\mathscr{A}$ contains the constant functions.

(iii) $\mathscr{A}$ separates point, i.e. for $x, y \in X$, there is some $f \in \mathscr{A}$ such that $f(x) \neq f(y)$.

(iv) $f \in \mathscr{A}$ implies $\overline{f} \in \mathscr{A}$.

Then $\mathscr{A}$ is dense in $C(X)$.

Proposition B.4. *(Proposition 2.51 of [EW17]) Let X be a locally compact σ-compact metric space equipped with a locally finite measure μ on the Borel σ-algebra. Then for any $1 \leq p < \infty$, $C_c(C)$ is dense in $L^p_\mu(X)$.*

Definition B.5. Let $L : V_1 \to V_2$ be a linear map also called a linear operator between two normed vector spaces $(V_1, ||\cdot||_{V_1})$ and $(V_2, ||\cdot||_{V_2})$. We define the operator norm of L to be

$$||L||_{\mathrm{op}} = \sup_{v \in V_1, ||v||_{V_1} \leq 1} ||Lv||_{V_2}.$$

We denote by $B(V_1, V_2)$ the space of linear operators with bounded operator norm. We also write $B(V_1)$ for the space $B(V_1, V_1)$.

Lemma B.6. *(Lemma 2.52 of [EW17]) A linear map between two normed vector spaces is continuous if and only if the operator norm is finite.*

Lemma B.7. *For V_1 and V_2 Banach spaces, the space $B(V_1, V_2)$ together with the operator norm forms a Banach space.*

Definition B.8. Let $\mathscr{A}$ be a Banach space and assume that there is a multiplication $\mathscr{A} \times \mathscr{A} \to \mathscr{A}, (x, y) \mapsto xy$ such that for all $x, y \in \mathscr{A}$,

$$||xy|| \leq ||x||\,||y||.$$

Then $\mathscr{A}$ is called a *Banach algebra*. A Banach algebra $\mathscr{A}$ is called *commutative* if $xy = yx$ for all $x, y \in \mathscr{A}$.

B.2 Hilbert Spaces

Definition B.9. A *pre-Hilbert space* is a vector space V over $\mathbb{R}$ (or $\mathbb{C}$) with an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ (or $\mathbb{C}$) such that

(i) $\langle v, v \rangle > 0$ for all $v \in V \backslash \{0\}$.

(ii) $\langle v, w \rangle = \overline{\langle v, w \rangle}$ for all $v, w \in V$.

(iii) The map $g \mapsto \pi_g v$ is continuous for any fixed $v \in V$.

Proposition B.10. *(Cauchy-Schwarz, Proposition 3.2 of [EW17]) Let $(V, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space. Then we have the Cauchy-Schwarz inequality,*

$$|\langle v, w \rangle| \leq ||v|| \, ||w||.$$

Definition B.11. A *Hilbert space* is a pre-Hilbert space $(V, \langle \cdot, \cdot \rangle)$ such that the induced norm

$$||v||_{\mathscr{H}} = \sqrt{\langle v, v \rangle}$$

is complete.

Definition B.12. Let $\mathscr{H}$ be a Hilbert space and $\mathscr{A} \subset \mathscr{H}$ be any subset. Then the orthogonal complement of A is defined to be

$$A^\perp = \{h \in \mathscr{H} \mid \langle h, a \rangle = 0 \text{ for all } a \in A\}.$$

Proposition B.13. *(Corollary 3.17 of [EW17]) Let $\mathscr{H}$ be a Hilbert space and let $Y \subset \mathscr{H}$ be a closed subspace. Then $Y^\perp$ is a closed subspace with*

$$\mathscr{H} = Y \oplus Y^\perp.$$

Proposition B.14. *(Corollary 3.18 of [EW17]) For a closed subspace Y of a Hilbert space $\mathscr{H}$, the orthogonal projection onto Y, defined by*

$$P_Y \mathscr{H} \longrightarrow Y, \qquad h \longmapsto y,$$

where y is the unique element of Y such that $h - y \in Y^\perp$, is a bounded linear operator with $||P_Y|| \leq 1$ satisfying and characterized by $\langle h, y \rangle = \langle P_Y h, y \rangle$ for all $h \in \mathscr{H}$ and $y \in Y$.

Definition B.15. We denote $V^* = B(V, \mathbb{R})$ or if V is a real vector space and $V^* = B(V, \mathbb{C})$ is V is a complex vector space. We call V^* the *dual space* of V. Note that V^* forms a Banach space equipped with the operator norm.

Theorem B.16. *(Fréchet-Riesz representation, Corollary 3.19 of [EW17]) For a Hilbert space $\mathscr{H}$ the map sending $h \in \mathscr{H}$ to $\phi(h) \in \mathscr{H}^*$ defined by $\phi(h)(x) = \langle x, h \rangle$ is a linear (respectively semi-linear in the complex case) isometric isomorphism between $\mathscr{H}$ and its dual $\mathscr{H}^*$*

Definition B.17. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and let $B : \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded operator. Then by the Fréchet-Riesz representation Theorem there exists a unique bounded operator $B^* : \mathscr{H}_2 \to \mathscr{H}_1$ such that

$$\langle Av_1, v_2 \rangle_{\mathscr{H}_1} = \langle v_1, Av_2 \rangle_{\mathscr{H}_2}$$

called the *adjoint* of B with $||B^*||_{\text{op}} = ||B||_{\text{op}}$.

Lemma B.18. *(Lax-Milgram Lemma, Exercise 3.21 of [EW17]) Let $\mathscr{H}$ be a Hilbert space and let $B : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ (or $\mathbb{C}$) is bilinear (or sesquilinear in the complex case) such that for all $x, y \in \mathscr{H}$,*

$$|B(x,y)| \leq M \cdot ||x|| \cdot ||y||$$

for $M > 0$. Then there exists a unique linear operator $T : \mathscr{H} \to \mathscr{H}$ with

$$B(x,y) = \langle Tx, y \rangle$$

for which $||T||_{\text{op}} \leq M$.

Proof. Fix $x \in \mathscr{H}$, then $B(x, \cdot) : \mathscr{H} \to \mathbb{R}$ (or $\mathbb{C}$) is a linear (or sesquilinear) form. Hence by the Fréchet-Riesz representation theorem there is a unique $x^* \in \mathscr{H}$ such that

$$B(x,y) = \langle x^*, y \rangle.$$

So denote by $T : \mathscr{H} \to \mathscr{H}$ the map that sends x to x^*. Note that this is not the trivial operator and hence $||T||_{\text{op}} > 0$. This map is a linear operator and we have by assumption

$$\begin{aligned} ||T||_{\text{op}}^2 &= \sup\nolimits_{x \in \mathscr{H}, ||x|| \leq 1} \langle Tx, Tx \rangle \\ &= \sup\nolimits_{x \in \mathscr{H}, ||x|| \leq 1} B(Tx, x) \\ &\leq \sup\nolimits_{x \in \mathscr{H}, ||x|| \leq 1} M \cdot ||Tx|| \cdot ||x|| \\ &\leq ||T||_{\text{op}} \cdot \sup\nolimits_{x \in \mathscr{H}, ||x|| \leq 1} M \cdot ||x|| \\ &\leq ||T||_{\text{op}} \cdot M. \end{aligned}$$

Hence $||T||_{\text{op}} \leq M$. □

Definition B.19. A finite or countable list (v_n) in a Hilbert space is called orthonormal if $\langle x_m, x_n \rangle = \delta_{mn} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$

Definition B.20. Let $\mathscr{H}_n$ be a finite or countable list of Hilbert spaces. Then we define the *direct Hilber space sum* as

$$\bigoplus_n \mathscr{H}_n := \{(v_n) \,:\, v_n \in \mathscr{H}_n \text{ and } \sum_n ||v_n||^2 < \infty\}.$$

This space forms a Hilbert space together with the inner product defined by

$$\langle (v_n), (w_n) \rangle_{\oplus} = \sum_n \langle v_n, w_n \rangle_{\mathscr{H}_n}.$$

Definition B.21. Let $\mathscr{H}$ be a Hilbert space. Then the *closed linear hull* of an orthonormal list (x_n) is given by

$$\begin{aligned} \overline{\langle \{x_n\} \rangle} &:= \left\{ \sum_n a_n x_n \,:\, \text{the sum converges in } \mathscr{H} \right\} \\ &:= \left\{ \sum_n a_n x_n \,:\, \sum_n |a_n|^2 < \infty \right\} \end{aligned}$$

Definition B.22. A list of orthonormal vectors in a Hilbert space is said to be *complete* or to be an *orthonormal basis* if its closed linear hull is $\mathscr{H}$.

Definition B.23. An operator between Hilbert spaces $U : \mathscr{H}_1 \to \mathscr{H}_2$ is called *unitary* if

$$\langle Uv_1, Uv_2\rangle_{\mathscr{H}_2} = \langle v_1, v_2\rangle_{\mathscr{H}_2}$$

for all $v_1, v_2 \in \mathscr{H}$.

Definition B.24. A unitary representation of a topological group on a Hilbert space $\mathscr{H}$ is a map $\pi : G \to B(\mathscr{H})$ written as $\pi(g)$ such that

(i) $\pi(e)$ is the identity operator.

(ii) $\pi(g_1)\pi(g_2) = \pi(g_1) \circ \pi(g_2) = \pi(g_1 g_2)$ for all $g_1, g_2 \in G$.

(iii) $\pi(g) : \mathscr{H} \to \mathscr{H}$ is a unitary operator.

(iv) $g \mapsto \pi_g v$ is continuous for all $v \in \mathscr{H}$.

Consider now a unimodular locally compact metric group G with Haar measure μ. The *regular right* representation is defined on $L^2_\mu(G)$ for $g \in G$,

$$(\pi(g)f)(x) = f(xg).$$

Proposition B.25. *The regular right representation is a unitary representation.*

Proof. Property (i) is clear. For (ii) consider $g_1, g_2 \in G$, then

$$(\pi(g_1)\pi(g_2)f)(x) = (\pi(g_1)f)(xg_2) = f(xg_1g_2) = (\pi(g_1g_2)f)(x).$$

To see (iii) let $g \in G$ and $f_1, f_2 \in L^2_\mu(G)$,

$$\begin{aligned}\langle \pi(g)f_1, \pi(g)f_2\rangle &= \int_G f_1(xg)\overline{f_2(xg)}\, d\mu(x) \\ &= \int_G f_1(x)\overline{f_2(x)}\, d\mu(x) = \langle f_1, f_2\rangle,\end{aligned}$$

where we used right invariance of the Haar measure for the second equal sign.

Lastly, to see (iv) it suffices to consider functions of compact support, since they form a dense subset in $L^2_\mu(G)$ by Proposition B.4. So let $f \in C_c(G)$. Then f is uniformly continuous. Hence there is some $\delta > 0$ such that

$$d(g, e) < \delta \quad \Rightarrow \quad |f(xg) - f(x)| < \frac{\varepsilon}{\sqrt{\mu(\mathrm{supp}(f))}}.$$

Thus

$$\begin{aligned}||\pi(g)f - f||_2^2 &= \int_G |f(xg) - f(x)|^2\, d\mu(g) \\ &\leq \int_K \frac{\varepsilon^2}{\mu(K)}\, d\mu(g) = \varepsilon^2.\end{aligned}$$

Hence if $g \to 0$, then $\pi(g)f \to 0$. Since we consider a group action, this implies continuity. $\square$

B.3 Compact, Self-Adjoint and Hilbert-Schmidt Operators

Definition B.26. For B_1 and B_2 are Banach spaces, we call a bounded linear operator $T : B_1 \to B_2$ *compact* if

$$\overline{T(U_1^{B_1})} \subset B_2$$

is compact in B_2, where $U_1^{B_1}$ denotes the unit ball in B_1.

Definition B.27. For $\mathscr{H}$ be a Hilbert space with orthonormal basis $e_0, e_1, e_2, \ldots$. An operator $T \in B(\mathscr{H})$ is called Hilbert-Schmidt if

$$\sum_{i,j} |\langle Te_j, e_i \rangle|^2 < \infty$$

Theorem B.28. *(Theorem 3.1.5 of [Zim90]) Let X be a compact metric space with a finite measure μ. If $K \in C(X \times X)$, then the integral operator*

$$(T_K f)(x) = \int_X K(x, y) f(y)\, d\mu(y)$$

defines a compact operator $T_K : L^2(X) \to L^2(X)$.

Theorem B.29. *(Proposition 3.1.12 of [Zim90]) Let (X, μ) be a measure space and $K \in L^2_\mu(X \times X)$, then the integral operator*

$$(T_K f)(x) = \int_X K(x, y) f(y)\, d\mu(y)$$

defines a Hilbert-Schmidt operator $T_K : L^2(X) \to L^2(X)$.

Definition B.30. A bounded operator $B : \mathscr{H} \to \mathscr{H}$ on a Hilbert space $\mathscr{H}$ is called self-adjoint if $B^* = B$.

Theorem B.31. *(Spectral Theorem for compact self-adjoint operators, Theorem 6.27 of [Zim90]) Let $\mathscr{H}$ be a separable and infinite-dimensional Hilbert space and let B be a compact, self-adjoint and bounded operator. Then there is an orthonormal basis (v_n) consisting of eigenvectors for B with real eigenvalues λ_n. Furthermore for each eigenvalue $\lambda_n \neq 0$, the eigenspace of λ_n is finite dimensional and we have*

$$\mathscr{H} = \bigoplus_n E_{\lambda_n},$$

there the direct sum is taken over all eigenvalues.

Definition B.32. Let $(X, \mathscr{B}, \mu)$ be a measure space, $\mathscr{H} = L^2_\mu(X)$ and let $g : X \to \mathbb{C}$ be a measurable function. Denote by M_g the multiplication operator given by

$$M_g : \mathscr{H} \to \mathscr{H}, \qquad f \mapsto gf.$$

Theorem B.33. *(Spectral Theorem for self-adjoint operators, Theorem 12.55 of [EW17]) Let $\mathscr{H}$ be a separable complex Hilbert space and $T \in B(\mathscr{H})$ a continuous self-adjoint operator. There there exists a finite measure space (X, μ), a unitary isomorphism*

$$\phi : \mathscr{H} \to L^2_\mu(X)$$

and a bounded measurable function $g : X \to \mathbb{R}$ such that

$$M_g \circ \phi = \phi \circ T,$$

where M_g is the multiplication operator given by $M_g : L^2_\mu(X) \to L^2_\mu(X), f \mapsto gf$.

B.4 Trace Class Operators

Definition B.34. Let $\mathscr{H}$ be a Hilbert space. A linear operator $A : \mathscr{H} \to \mathscr{H}$ is called *trace-class* if its *trace-class norm*

$$||A||_{\mathrm{tc}} = \sup_{(v_n),(w_n)} \sum_{n=1}^{N} |\langle Av_n, w_n\rangle|$$

is finite, where the supremum is taken over all integers $N \geq 0$ and over any two finite lists of orthonormal vectors $(v_1, \dots, v_N)$ and $(w_1, \dots, w_M)$ of the same length N.

We denote by $\mathrm{TC}(\mathscr{H})$ the space of trace-class operators. Note that every trace class operator is a bounded operator and hence $\mathrm{TC}(\mathscr{H}) \subset B(\mathscr{H})$.

Theorem B.35. *(Theorem 6.39 of [EW17]) Let $\mathscr{H}$ be a separable complex Hilbert space. Then there exists a linear functional $\mathrm{tr} : \mathrm{TC}(\mathscr{H}) \to \mathbb{C}$ with the following properties:*

1. *(i)* $|\mathrm{tr}(A)| \leq ||A||_{\mathrm{tc}}$,
2. *(ii)* $\mathrm{tr}(A) = \mathrm{tr}(U^{-1}AU)$, *and*
3. *(iii)* $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

for all $A \in \mathrm{TC}(\mathscr{H}), B \in \mathrm{B}(\mathscr{H})$ and unitary $U \in \mathrm{B}(\mathscr{H})$. Moreover,

1. *(iv)* $\mathrm{tr}(A) = \sum_{n=1}^{\infty}\langle Av_n, v_n\rangle$

for any $A \in \mathrm{TC}(\mathscr{H})$ and orthonormal basis (v_n) of $\mathscr{H}$.

Proposition B.36. *(Proposition 6.42 of [EW17]) Every trace-class operator on a complex Hilbert space $\mathscr{H}$ is compact.*

Proposition B.37. *(Proposition 6.44 of [EW17]) Let $\mathscr{H}$ be a complex Hilbert space and A a bounded operator on $\mathscr{H}$. If A is self-adjoint and positive and (v_n) is an orthonormal basis of $\mathscr{H}$, then $||A||_{\mathrm{tc}} = \sum_{n=1}^{\infty}\langle Av_n, v_n\rangle$.*

Theorem B.38. *(Corollary 6.46 of [EW17]) Let $\mathscr{H}$ be a separable complex Hilbert space. A self-adjoint bounded operator A on $\mathscr{H}$ is trace-class if and only if it is compact and its eigenvalues λ_n satisfy*

$$\sum_{n=1}^{\infty} |\lambda_n| = ||A||_{\mathrm{tc}} < \infty.$$

If A is indeed trace-class, then

$$\mathrm{tr}(A) = \sum_{n=1}^{\infty} \lambda_n$$

and the sum converges absolutely.

Proposition B.39. *(Proposition 6.48 of [EW17]) Let X be a compact metric space, let μ be a finite measure on X and let $k \in C(X \times X)$ be a continuous kernel with the property that the associated Hilbert Schmidt operator K defined by*

$$K(f)(x) = \int_X k(x,y) f(y)\, d\mu(y)$$

is trace-class. Then

$$\operatorname{tr}(K) = \int_K k(x,x)\, d\mu(x).$$

B.5 Fourier Transform and Schwartz Space

Definition B.40. Let $f : \mathbb{R} \to \mathbb{C}$ be a function. We define the *Fourier transform* as

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{2\pi i x t}\, dx$$

and the inverse of the Fourier transform as

$$\check{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x t}\, dx.$$

Example B.41. Consider $f(x) = e^{-2\pi r|x|}$ for $r \geq 0$ a positive parameter. Then

$$\begin{aligned}
\hat{f}(x) &= \int_{-\infty}^{\infty} f(t) e^{2\pi i t x}\, dt \\
&= \int_{0}^{\infty} e^{2\pi t(ix - r)} dt + \int_{-\infty}^{\infty} e^{2\pi t(ix+r)}\, dt \\
&= \frac{1}{2\pi}\left[\frac{1}{ix - r} e^{t(2\pi i x - r)}\right]_0^{\infty} + \frac{1}{2\pi}\left[\frac{1}{ix+r} e^{t(2\pi i x + r)}\right]_{-\infty}^{0} \\
&= \frac{1}{2\pi}\left(\frac{1}{ix+r} - \frac{1}{ix-r}\right) \\
&= \frac{1}{2\pi}\frac{r}{x^2+r^2}.
\end{aligned}$$

Proposition B.42. *(Proposition 9.34 of [EW17]) The Fourier transform maps $L^1(\mathbb{R})$ into $C_0(\mathbb{R})$.*

Theorem B.43. *(Theorem 9.36 of [EW17]) If $f \in L^1(\mathbb{R})$ has $\hat{f} \in L^1(\mathbb{R})$, then f agrees almost everywhere with the continuous function $(f)^{\wedge\vee} \in C_c(\mathbb{R})$.*

Corollary B.44. *If $f, \hat{f} \in L^1(\mathbb{R})$ then $f^{\wedge\wedge}(x) = f(-x)$ almost everywhere.*

Proof. We have almost everywhere

$$f^{\wedge\wedge}(x) = \int_{-\infty}^{\infty} \hat{f}(t) e^{2\pi i t x}\, dt = \int_{-\infty}^{\infty} \hat{f}(t) e^{2\pi i t x}\, dt = f^{\wedge\vee}(-x) = f(-x).$$

$\square$

Theorem B.45. *If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\hat{f}$ lies in $L^2(\mathbb{R})$ with $||\hat{f}||_2 = ||f||_2$ and the map $f \mapsto \hat{f}$ extends continuously to a unitary operator on $L^2(\mathbb{R})$.*

Definition B.46. The *Schwartz Space* on $\mathbb{R}$ is defined by

$$\mathscr{S}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ is smooth and } ||x^\alpha \partial_\beta f||_\infty < \infty \text{ for all } \alpha, \beta \in \mathbb{N}_0\}.$$

Theorem B.47. *The Fourier transform maps $\mathscr{S}(\mathbb{R})$ to itself, is a continuous operator and has the Fourier back transform as inverse.*

Theorem B.48. *(Poisson summation formula) For $f \in \mathscr{S}(\mathbb{R})$ we have*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

References

[Bea91] Alan Beardon. *The Geometry of Discrete Groups*. Springer, 1991.

[Ber11] Nicolas Bergeron. *The Spectrum of Hyperbolic Surfaces*. Springer, 2011.

[Bro16] David Brothwick. *Spectral Theory of Infinite-Area Hyperbolic Surfaces*. Birkhäuser, 2016.

[Bum98] Daniel Bump. *Automorphic Forms and Representations*. Cambridge Studies in Advanced Mathematics, 1998.

[Bur07] Marc Burger. *Notes for the Selberg's Trace Formula*. 2007.

[Bur16] Marc Burger. *Lecture Notes for Riemannian Geometry*. 2016.

[Bus92] Peter Buser. *Geometry and Spectra of Compact Riemann Surfaces*. Birkhäuser, 1992.

[EW11] Manfred Einsiedler and Thomas Ward. *Ergodic Theory with a view towards Number Theory*. Springer, 2011.

[EW17] Manfred Einsiedler and Thomas Ward. *Functional Analysis, Spectral Theory and Applications*. Springer, 2017.

[Far83] Jacques Faraut. In *Analyse harmonique*, chapter 4, pages 315–446. CIMPA, 1983.

[GGPS69] I. Gelfand, M. Graev, and I. Pyatetskii-Shapiro. *Representation Theory and Automorphic Functions*. W.B. Saunders Company, 1969.

[Hel01] Sigurdur Helgason. *Differential Geometry, Lie Groups and Symmetric Spaces*. American Mathematical Society, 2001.

[Hel08] Sigurdur Helgason. *Geometric Analysis on Symmetric Spaces: Second Edition*. American Mathematical Society, 2008.

[Ioz17] Alessandra Iozzi. *Lecture Notes for Symmetric Spaces*. 2017.

[IZ17] Alessandra Iozzi and Robert Zimmer. *Lecture Notes for Lie Groups*. 2017.

[Kri13] Yannick Krifka. *Fenchel Nielsen Koordinaten auf Teichmüllerräumen*. Bachelorthesis, January 2013.

[Zim90] Robert Zimmer. *Essential Results of Functional Analysis*. Chicago Lectures in Mathematics, 1990.