QUADRATIC FORMS AND DUKE’S THEOREM

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Abstract. These notes are a continuation of [Kog19] and we relate quadratic forms to the collection of curves discussed in [Kog19]. We discuss content from Section 3 of the 2012 paper [ELMV12] by Einsiedler, Lindenstrauss, Michel and Venkatesh.

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1. Introduction

We study binary integral quadratic forms

$$Q(X, Y) = aX^2 + bXY + cY^2$$

for $a, b, c \in \mathbb{Z}$. For $X = (X, Y)^T$, the above quadratic form can be rewritten as

$$Q(X, Y) = X^T B_Q X \quad \text{for} \quad B_Q = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}.$$ 

Moreover, the invariant $d_Q = b^2 - 4ab = -4\det(B_Q)$ is called the discriminant of $Q$. We note that only the integers $\equiv 0, 1 \mod 4$ can be discriminants.

In these notes only primitive quadratic forms are considered, i.e. it is assumed that $\gcd(a, b, c) = 1$. Moreover, we will not distinguish between the quadratic form $Q(X, Y)$ and the tuple $(a, b, c) \in \mathbb{Z}^3$ it defines. We write $Q_d$ for the set of primitive quadratic forms and only the case where $d$ is positive and not a perfect square is studied.

Recall that $X = \Gamma \backslash G = \text{PGL}_2(\mathbb{Z}) \backslash \text{PGL}_2(\mathbb{R})$ can be viewed as the space of lattices up to scaling. We denote by $A$ the diagonal subgroup of $\text{PGL}_2(\mathbb{R})$.

As $d$ is a discriminant, we have that $O_d := \mathbb{Z}\left\lceil \frac{d+\sqrt{d}}{2} \right\rceil \subset K := \mathbb{Q}(\sqrt{d})$ is an
order in $K$. Recall that
\[ \text{Pic}(\mathcal{O}_d) = \{ \text{invertible fractional } \mathcal{O}_d\text{-ideal of } K \}/K^\times. \]

In [Kog19] we related to each invertible fractional $\mathcal{O}_d$-ideal of $K$ a periodic $A$-orbit $x_a.A$, which only depends on the equivalence class $[a] \in \text{Pic}(\mathcal{O}_d)$. This allows us to define
\[ \mathcal{G}_d = \bigcup_{[a] \in \text{Pic}(\mathcal{O}_d)} x_a.A. \tag{1.1} \]

The set $\mathcal{G}_d$ supports a canonical $A$-invariant measure, which we denote by $\mu_d$. A first aim of these notes is to relate the collection of periodic $A$-orbits $\mathcal{G}_d$ to quadratic forms. More precisely, we will describe $\mathcal{G}_d$ in terms of information given by quadratic forms. This will be done in Section 3.

Recall that the height of a lattice $L$ is defined as
\[ \text{ht}(L) = \left( \frac{\min_{v \in L \setminus \{0\}} \|v\|}{\text{vol}(L)^{1/2}} \right)^{-1}. \]

We write $X_{\geq H} = \{ x \in X : \text{ht}(x) \geq H \}$ and $X_{< H} = \{ x \in X : \text{ht}(x) < H \}$. Proposition 3.3. of [ELMV12] (and Proposition 7.4 of [Kog19]) implies that for any $\varepsilon > 0$,
\[ \mu_d(X_{\geq \varepsilon}) \to 0 \]
and $d \to \infty$.

With the help the viewpoint on $\mathcal{G}_d$ given by quadratic forms, we will be able to estimate how many of the points $(x_1, x_2) \in (\mathcal{G}_d \cap X_{\leq H})^2$ are $\delta$-close. To make this more precise, we next discuss how to define a metric on $X$. First recall that there is a left-invariant metric $d_G$ on $G$ (see Section 9.3 of [EW11]), i.e. a metric that induces the topology of $G$ and satisfies $d_G(hg_1, hg_2) = d_G(g_1, g_2)$ for any $g_1, g_2, h \in G$. This metric induces a well-defined metric on $X = \Gamma \backslash G$ given as
\[ d_X(\Gamma g_1, \Gamma g_2) = \inf_{\gamma \in \Gamma} d_G(g_1, \gamma g_2). \]

We now can state the main proposition of these notes.

**Proposition 1.1.** (Main Proposition, Proposition 3.6. of [ELMV12]) For any $\varepsilon > 0$ and $H$ large we have
\[ (\mu_d \times \mu_d) \left( \{ (x, y) \in X_{< H} \times X_{< H} : d_X(x, y) < \delta \} \right) \ll \varepsilon H^4 \delta^3 d^\varepsilon, \]
where $d^{-1/4} \leq \delta \leq H^{-2}$.

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2. Quadratic Forms and the Picard Group

For any \( g \in \mathrm{GL}_2(\mathbb{R}) \) and any binary quadratic form \( Q \) (not necessarily an element of \( Q_d \)) we define
\[
(g.Q)(X,Y) = \frac{1}{\det(g)}Q((X,Y)g)
\]
and we note that
\[
B_{g,Q} = \frac{1}{\det(g)}gB_Qg^T.
\]
So we see that \( g.Q \) has the same discriminant as \( Q \). Thus \( \mathrm{GL}_2(\mathbb{R}) \) acts on the space of all quadratic forms of discriminant \( d \) and the action factors through \( \mathrm{PGL}_2(\mathbb{R}) \).

As before we write \( Q_d \) for the set of primitive integral binary quadratic forms of discriminant \( d > 0 \), for \( d \) not a perfect square. If \( Q \in Q_d \), then it is not guaranteed that \( g.Q \in Q_d \) for a general element \( g \in \mathrm{GL}_2(\mathbb{R}) \). However \( \mathrm{PGL}_2(\mathbb{Z}) \) defines a well defined action on \( Q_d \).

We say that two quadratic forms \( Q_1, Q_2 \in Q_d \) are equivalent if their \( \mathrm{PGL}_2(\mathbb{Z}) \)-orbits are the same. In particular, this means that \( Q_1 \) and \( Q_2 \) are equivalent if there is some \( g \in \mathrm{PGL}_2(\mathbb{Z}) \) such that \( Q_1 = g.Q_2 \). It is clear that this defines an equivalence relation and we denote by \( [Q_d] \) the set of equivalence classes.

**Theorem 2.1.** The set \( [Q_d] \) has the same cardinality as the Picard group \( \mathrm{Pic}(\mathcal{O}_d) \).

For a proof of Theorem 2.1 we refer to [Kog]. In these notes we only construct a map from \( \mathrm{Pic}(\mathcal{O}_d) \to [Q_d] \). Namely, for each invertible fractional \( \mathcal{O}_d \)-ideal \( a \) and an integral basis \( a_1, a_2 \) of \( a \) we define
\[
Q_{a_1,a_2}(X,Y) = \frac{N(a_1X + a_2Y)}{N(a)}.
\]
Furthermore, \( Q_{a_1,a_2} \) has discriminant \( d \) as
\[
d_{Q_{a_1,a_2}} = \frac{1}{N(a)^2}((a_1\sigma a_2 + \sigma a_1a_2)^2 - 4a_1\sigma a_1\sigma a_1a_2)
= \frac{1}{N(a)^2}(a_1\sigma a_2 - \sigma a_1a_2)^2
= \frac{1}{N(a)^2} \det \left( \begin{array}{cc} a_1 & a_2 \\ \sigma a_1 & \sigma a_2 \end{array} \right)^2 = \frac{d(a)}{N(a)^2} = d
\]
Finally, \( Q_{a_1,a_2} \) as an element of \( [Q_d] \) does not depend on the choice of integral basis and so defines an element \( Q_a \in [Q_d] \). Moreover, \( Q_a \) only depends on the ideal class \( [a] \in \mathrm{Pic}(\mathcal{O}_d) \). So we have constructed a map
\[
\mathrm{Pic}(\mathcal{O}_d) \to [Q_d], \quad [a] \mapsto [Q_a],
\]
which is in fact a bijection.
3. \(A\)-orbits associates to Quadratic Forms

Write \(\mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}\). For \((a, b, c) \in \mathbb{Q}_d\) we set
\[
x_{a,b,c,\pm} = \frac{-b + \sqrt{d}}{2c}, \in \mathbb{R} = \partial \mathbb{H}.
\]
These two points uniquely determine a geodesic on \(\mathbb{H}\) and also a lift onto the unit tangent bundle \(T^1(\mathbb{H}) \cong \text{SL}(\mathbb{R})\) which we call \(\gamma_{(a,b,c)}\). Denote by \(Q_0\) the quadratic form \(Q_0(X,Y) = XY\). We claim that
\[
\gamma_{(a,b,c)} = \{ g \in \text{PGL}_2(\mathbb{R}) : \sqrt{d}(g.Q_0)(X,Y) = aX^2 + bXY + cY^2 \}. \tag{3.1}
\]
To see this we denote \(h_{a,b,c} = (b + \sqrt{d} \ b - \sqrt{d})/2c\) and \(w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \text{SL}(\mathbb{Z})\).

Then \(wh_{a,b,c} = (2c \ b + \sqrt{d})/b - \sqrt{d}\) and so
\[
wh_{a,b,c,0} = -\frac{2c}{b - \sqrt{d}} = -\frac{2c}{b - \sqrt{d}} \frac{b + \sqrt{d}}{b + \sqrt{d}} = -\frac{2cb - 2c\sqrt{d}}{-4ac} = \frac{b + \sqrt{d}}{2a}
\]
and
\[
wh_{a,b,c,\infty} = -\frac{2c}{b + \sqrt{d}} = -\frac{2c}{b + \sqrt{d}} \frac{b - \sqrt{d}}{b - \sqrt{d}} = -\frac{2cb + 2c\sqrt{d}}{-4ac} = \frac{b - \sqrt{d}}{2a}.
\]
Furthermore we calculate
\[
\frac{1}{\det(h_{a,b,c})}h_{a,b,c} \begin{pmatrix} 0 & 1 \\ 1/2 & 0 \end{pmatrix} h_{a,b,c}^T = \frac{1}{4c\sqrt{d}} \begin{pmatrix} b^2 - d & 2cb \\ 2cb & 4c^2 \end{pmatrix} = \frac{1}{\sqrt{d}} \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}
\]
and this shows that
\[
\sqrt{d}(h_{a,b,c}.Q_0)(X,Y) = aX^2 + bXY + cY^2.
\]
Hence (3.1) follows as \(A\)-is the stabilizer of \(Q_0\).

We can project \(\gamma_{(a,b,c)}\) to the curve \(\gamma_{(a,b,c)}^X\) on \(X = T^1(\text{SL}(\mathbb{Z})\setminus\mathbb{H}) \cong \text{PGL}_2(\mathbb{Z})\setminus\text{PGL}_2(\mathbb{R})\). If \((a, b, c) \in \mathbb{Q}_d\) and \((a', b', c') \in \mathbb{Q}_d\) define the same \(\text{PGL}_2(\mathbb{Z})\) equivalence class, then \(x_{a,b,c,\pm}\) and \(x_{a',b',c',\pm}\) lie on the same \(\text{PGL}_2(\mathbb{Z})\) orbit. Thus \(\gamma_{(a,b,c)}^X\) only depends on the class \([a, b, c] \in \mathbb{Q}_d\). Thus we can define
\[
G_d = \bigcup_{[(a,b,c)] \in \mathbb{Q}_d} \gamma_{[(a,b,c)]}^X.
\]
This definition agrees with (1.1), see [ELMV12] section 2.3 for a discussion why this is the case and [Kog] section 4.4. for a detailed proof.
4. Representations of Quadratic Forms

Let $q$ be an integral quadratic form in $n$-variables and let $Q$ be one in $m$-variables, where we assume $n \leq m$. We call a $\mathbb{Z}$–linear map $\iota : \mathbb{Z}^n \to \mathbb{Z}^m$ a representation of $q$ by $Q$ if for all $x \in \mathbb{Z}^n$ we have

$$Q(\iota(x)) = q(x).$$

Denote by $R_Q(q)$ the set of such representations. We write

$$\text{SO}_Q(\mathbb{Z}) = \{A \in M_m(\mathbb{Z}) : Q(Ax) = Q(x) \text{ for all } x \in \mathbb{Z}^m\}$$

for the special orthogonal group with respect to $Q$. Then $\text{SO}_Q$ naturally acts on the set $R_Q(q)$ and the quotient $\text{SO}_Q(\mathbb{Z}) \backslash R_Q(q)$ is finite.

For a more extended discussion concerning the representation of integral quadratic forms we refer to Section 3.2 of [ELMV12]. We will be only interested in the representation of the quadratic form $q(X, Y) = dX^2 + \ell XY + dY^2$ for $d$ as above and $\ell$ some integer by the ternary form given by the discriminant $\text{disc}(X, Y, Z) = Y^2 - 4XZ$. In this situation the following result holds.

**Corollary 4.1.** (Corollary 3.5 of [ELMV12]) Assume that $\ell \neq \pm 2d$, then

$$|\text{SO}_{\text{disc}}(\mathbb{Z}) \setminus \{(\mathbb{Z}^2, dX^2 + \ell XY + dY^2) \mapsto (\mathbb{Z}^3, \text{disc})\}| \ll_{\varepsilon} f \max(|d|, |\ell|)^{\varepsilon}$$

where $f^2$ is the largest square divisor of $\gcd(d, \ell)$.

Moreover, we next explain how we can embed $\text{PGL}_2(\mathbb{Z})$ into $\text{SO}_{\text{disc}}(\mathbb{Z})$ with finite index. Denote

$$v = v(a, b, c) = \begin{pmatrix} b & 2a \\ -2c & -b \end{pmatrix},$$

so that we can write $\mathfrak{sl}_2(\mathbb{R}) = \{v(a, b, c) : a, b, c \in \mathbb{R}\}$ and note that $\mathfrak{sl}_2(\mathbb{R})$ is a three dimensional vector space over $\mathbb{R}$ on which we define a quadratic form

$$Q(v(a, b, c)) = \det(v) = -b^2 + 4ac = -\text{disc}(a, b, c).$$

We now view $\mathbb{R}^3 \cong \mathfrak{sl}_2(\mathbb{R})$ and so $(\mathbb{R}^3, Q)$ is a quadratic space. Observe that $\text{PGL}_2(\mathbb{R})$ acts on $\mathfrak{sl}_2(\mathbb{R}) \cong \mathbb{R}^3$ via $g.v = gvg^{-1}$ for $g \in \text{PGL}_2(\mathbb{R})$ and $v \in \mathfrak{sl}_2(\mathbb{R}) \cong \mathbb{R}^3$ and moreover $Q(g.v) = \det(g.v) = \det(v) = Q(v)$. This shows that this action defines an injective representation

$$\text{PGL}_2(\mathbb{R}) \hookrightarrow \text{SO}_Q(\mathbb{R}), \quad g \mapsto (v \mapsto gvg^{-1})$$

where injectivity follows easily. Thus $\text{PGL}_2(\mathbb{R})$ is isomorphic to either $\text{SO}_Q(\mathbb{R})$ or to $\text{SO}_Q(\mathbb{R})^\circ$. Now note

$$\text{SO}_Q(\mathbb{Z}) = \{g \in \text{SO}_Q(\mathbb{R}) : g.\mathfrak{sl}_2(\mathbb{Z}) \subset \mathfrak{sl}_2(\mathbb{Z})\}$$

and so in particular $\text{PGL}_2(\mathbb{Z})$ can we viewed as a subset of $\text{SO}_Q(\mathbb{Z})$. As $\text{PGL}_2(\mathbb{R})$ can be viewed as a finite index subgroup of $\text{SO}_Q(\mathbb{R})$ it follows that $\text{PGL}_2(\mathbb{Z})$ is a sub-lattice of $\text{SO}_Q(\mathbb{Z})$ and thus has finite index.
This shows that we can replace in the statement of 4.1 the group SO_{disc}(\mathbb{Z}) by the image of SL_2(\mathbb{Z}) under this injection. This implies
$$|SL_2(\mathbb{Z}) \setminus \{(\mathbb{Z}^2, dX^2 + \ell XY + dY^2) \leftrightarrow (\mathbb{Z}^3, \text{disc})\}| \ll_{\varepsilon} f \max(|d|, |\ell|)^{\varepsilon}. \quad (4.1)$$

5. Proof of Proposition 1.1

Denote by $\mathcal{F}$ the fundamental domain of $X$ given by
$$\mathcal{F} = \{(z, v) \in \mathbb{H} \times S^1 \text{ such that } |\text{Re}(z)| \leq \frac{1}{2} \text{ and } |z| \geq 1\}$$

and by $\mathcal{F}'$ a slight extension of $\mathcal{F}$ defined as
$$\mathcal{F}' = \{(z, v) \in \mathbb{H} \times S^1 \text{ such that } |\text{Re}(z)| \leq 1 \text{ and } |z| \geq \frac{1}{2}\}.$$

We progress with a few preliminary observations. Let $x_1, x_2 \in X_{\leq H}$ such that $d_X(x_1, x_2) < \delta$. Write $x_i = \Gamma g_i$ for $i = 1, 2$ and $g_i \in \text{PGL}_2(\mathbb{R})$. In order to bound the coefficients of $g_i$, we always assume that the matrix $g_i$ has determinant $\pm 1$.

We choose $g_1$ such that $g_1 \in \mathcal{F}$ and $g_2 \in \mathcal{F}'$ such that $d_G(g_1, g_2) < \delta$. We claim that $||g_i|| \ll H$. We assume without loss of generality that $\det(g_1) = 1$. We use the NAK decomposition of SL_2(\mathbb{R}) in order to write
$$g_1 = \begin{pmatrix} a & t \\ 0 & a^{-1} \end{pmatrix} k,$$

for $a, t \in \mathbb{R}$ with $a \neq 0$ and for $k \in \text{SO}_2(\mathbb{R})$. As $g_1 \in \mathcal{F} \cap X_{\leq H}$ we conclude that $|\text{Re}(g_1. i)| \leq 1$ and $\frac{1}{2} \leq |\text{Im}(g_1. i)| \leq H^2$. We note that
$$g_1. i = a^2 i + at$$

and so $\frac{1}{2} \leq |a| \leq H$ and $|t| \leq \frac{1}{|a|} \leq 2$. Thus all the coefficients of $g_1$ are $\ll H$. As $g_2$ is close to $g_1$ we conclude that $||g_i|| \ll H$.

We now associate to $g_i$ the primitive integral quadratic form
$$q_i(X, Y) = \sqrt{d}[g_i. q_0](X, Y) = a_i X^2 + b_i XY + c_i Y^2$$

with $d = b_i^2 - 4a_i c_i$ and $\gcd(a_i, b_i, c_i) = 1$. Towards the estimate of Proposition 1.1 the case where $q_1 = q_2$ will be easier, so we focus on the case $q_1 \neq q_2$. We want to count the number of such possible tuples $g_1, g_2$ such that $q_1 \neq q_2$. The $\Gamma$-equivalence class of $(q_1, q_2)$ only depends on the point $x_1, x_2$ or more precisely by the $\delta$-neighborhood around $x_1$ and $x_2$. By compactness of $\mathcal{G}_d$ the number of distinct such quadratic form is finite and we write
$$\Gamma(q_1^{(1)}, q_2^{(1)}), \ldots, \Gamma(q_1^{(k)}, q_2^{(k)})$$

for a complete list of such quadratic forms. Our first aim is to count $k$ effectively.

Lemma 5.1. In the above setting,
$$k \ll_{\varepsilon} d^{1+2\varepsilon} H^4 \delta^2.$$
Proof. As \(|g_i| \ll H\), it follows that
\[
\max(|a_i|, |b_i|, |c_i|) \ll d^{1/2}H^2
\]
and as by assumption \(g_2 = g_1 h\) for \(d(h, \text{id}) < \delta\), we conclude that \(g_2 = \sqrt{d} g_1 (h, q_0)\) with \(\|h \cdot q_0 - q_0\| \ll \delta\) and so
\[
\max(|a_1 - a_2|, |b_1 - b_2|, |c_1 - c_2|) \ll d^{1/2}H^2\delta. \tag{5.1}
\]
We now define the quadratic form
\[
q(X, Y) = \text{disc}(X(a_1, b_1, c_1) + Y(a_2, b_2, c_2)) = dX^2 + \ell XY + dY^2
\]
for \(\ell \in \mathbb{Z}\). Thus the map
\[
\ell : \mathbb{Z}^2 \to \mathbb{Z}^3, \quad \begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}
\]
defines a representation of \(dX^2 + \ell XY + dY^2\) by the ternary form \(\text{disc}\).

Note that
\[
|2d - \ell| = |q(1, -1)| = \text{disc}(a_1 - a_2, b_1 - b_2, c_1 - c_2) \ll dH^4\delta^2
\]
and so there is only a finite number of possible values for \(\ell\). Furthermore, assuming that \(q_1 \neq q_2\) we show that \(\ell \neq \pm 2d\). Indeed, if \(\ell = \pm 2d\), then
\[
d(a_2 \mp a_1)^2 = da_2^2 \mp 2da_2a_1 + da_1^2 = q(a_2, -a_1)
\]
\[
= \text{disc}(a_2(a_1b_1c_1) - a_1(a_2, b_2, c_2)) = (a_2b_1 - a_1b_2)^2
\]
which contradicts the assumption that \(d\) is not a perfect square. So by \([4.1]\), \(N_{\ell,d} := |\text{SL}_2(\mathbb{Z})\{\{(\mathbb{Z}^2, dX^2 + \ell XY + dY^2) \leftrightarrow (\mathbb{Z}^3, \text{disc})\}| \ll \varepsilon \max(|d|, |\ell|)^\varepsilon
\]
and so \(N_{\ell,d} \ll f^d\varepsilon^d\) as \(d \geq 0\) and as by \([5.1]\)
\[
|\ell| \ll |\ell - 2d| + 2d \ll 2d + dH^4\delta^2 \ll d
\]
as \(d^{-1/4} \leq \delta \leq H^{-2}\).

If \(\Gamma(q_1(i), q_2(i))\) and \(\Gamma(q_1(j), q_2(j))\) are different then they define different embeddings up to \(\text{SL}_2(\mathbb{Z})\)-equivalence, where we view \(\text{SL}_2(\mathbb{Z}) \leftrightarrow \text{SO}_{\text{disc}}(\mathbb{Z})\). Thus
\[
k \leq \sum_{\text{all possible } \ell} N_{\ell,d}
\]
\[
\leq \sum_{f^2 \mid d} \sum_{\text{all possible } \ell} f^d\varepsilon
\]
\[
\ll \sum_{f^2 \mid d} f^d\varepsilon \frac{dH^4\delta^2}{f^2}
\]
\[
\ll \sum_{f^2 \mid d} d^{1+\varepsilon} H^4\delta^2
\]
\[
\ll \varepsilon d^{1+2\varepsilon} H^4\delta^2,
\]
where in the third line we used that $\ell$ has to satisfy $|2d - \ell| \ll dH^4\delta^2$ and $f^2|\ell|$ so that $\text{gcd}(\ell, \delta)$ is square-free. In the last line we used that the number of divisors of $d$ can be bounded by $\ll \varepsilon \delta^t$ for any $\varepsilon > 0$. \hfill $\square$

**Lemma 5.2.** Let $(x_1, x_2) = (\Gamma g_1, \Gamma g_2) \in (G_d \cap X_{\leq H})^2$ be as above such that $d_X(x_1, x_2) < \delta$ and $q_1 \neq q_2$. Then there is some $j$ so that $x_1 = \Gamma g_1^{(j)} a_t$ where $t \in I_j$ for $I_j$ some interval of length $\ll \log(d)$.

**Proof.** Choose $j$ so that $(\sqrt{d}(g_1.q.0), \sqrt{d}(g_2.q.0)) = (q_1^{(j)}, q_2^{(j)})$. We note that $G_d \subset X_{<d^{1/4}}$ and so using [5.1] there is a constant $c$ so that

$$\max(|a_1 - a_2|, |b_1 - b_2|, |c_1 - c_2|) \leq cd^{1/2}(d^{1/4})^2 \leq c\delta.$$

Thus $d(g_1 a_t, g_2 a_t') \geq \frac{1}{2c}d^{-1}$. In particular, $d(g_1 a_t, g_2 A) \geq \frac{1}{2c}d^{-1}$, so

$$d(g_1 a_t, g_2 A) \gg d^{-1}. \quad (5.2)$$

We now want to show that the inequality $d_G(g_1^{(j)} a_t, g_2^{(j)} A) < 1$ can only hold for $t$ in an interval of length $\ll \log(d)$. To see this denote by $h = (g_1^{(j)})^{-1} g_2$ and we write

$$h = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} \in \text{SL}_2(\mathbb{R})$$

with $||h|| \ll \delta$. Using left invariance of the metric, (5.2) is equivalent to

$$d_G \left( \begin{pmatrix} \pm e^t & 0 \\ 0 & \pm e^{-t} \end{pmatrix}, \begin{pmatrix} \pm e^s h_1 & \pm e^{-s} h_2 \\ \pm e^s h_3 & \pm e^{-s} h_4 \end{pmatrix} \right) \gg d^{-1} \quad (5.3)$$

for all $s, t \in \mathbb{R}$. We first show that $|h_2|, |h_3| \gg d^{-1}$. For a contradiction assume without loss of generality that $|h_2| \leq cd^{-1}$ for some small $c$ determined later, then we have that

$$|h_4| = \frac{|1 + h_2 h_3|}{|h_1|} \ll \frac{1 + cd^{-1}\delta}{|h_1|}.$$

We can assume that $h_1$ and $h_4$ are positive. For $s = 0$ in (5.3), we choose $t$ so that $e^t = h_1$. Then we can choose $c$ small enough so that

$$|e^{-t} - h_4^t| = \frac{|cd^{-1}\delta|}{|h_1|} \ll d^{-1}$$

and so we arrive at a contradiction to (5.3). Thus $|h_2|, |h_3| \gg d^{-1}$.

Assume that $s, t \in \mathbb{R}$ so that

$$d_G \left( \begin{pmatrix} \pm e^t & 0 \\ 0 & \pm e^{-t} \end{pmatrix}, \begin{pmatrix} \pm e^s h_1 & \pm e^{-s} h_2 \\ \pm e^s h_3 & \pm e^{-s} h_4 \end{pmatrix} \right) < 1 \quad (5.4)$$

Then we have that $|e^s h_2| \ll 1$ and $|e^{-s} h_3| \ll 1$ and so in particular $e^s \ll \frac{1}{|h_2|} \ll d$ and $e^{-s} \ll \frac{1}{|h_3|} \ll d$ and so we have that $|s| \ll \log(d)$. So we already see that $s$ can only be inside an interval of length $\ll \log(d)$. So we have that $|\pm e^t \mp e^s h_1| \ll 1$ and so $|e^t| \ll 1 + |e^s h_1| \ll 1 + \log(d)\delta \ll \log(d)$
for $H$ and so $d$ large enough (and in particular $\delta$ small). The same again holds for $e^{-t}$ and so we conclude $|t| \ll \log(d)$.

Proof. (of Proposition 1.1) Let $(x_1, x_2) = (\Gamma g_1, \Gamma g_2) \in (G_d \cap X \leq H)^2$ be such that $d_X(x_1, x_2) < \delta$ for $g_1, g_2$ as above. Then as before we associate to $x_i$ the quadratic form $q_i = \sqrt{d}(g_i, q_0)(X, Y)$. Recall that $\text{length}(G_d^{2}) = d^{1+o(1)}$. So it suffices to show that

$$\text{length}\left\{ (x, y) \in (G_d \cap X \leq H)^2 : d_X(x, y) < \delta \right\} \ll_{\varepsilon} H^4 \delta^3 d^{1+\varepsilon}$$

If $q_1 = q_2$ then $x_1$ and $x_2$ lie on the same geodesic and so in this case we have that all these points can be described by

$$\{(x, x_{at}) \in (G_d \cap X \leq H)^2 : |t| \ll \delta\} \subset \{(x, x_{at}) \in (G_d)^2 : |t| \ll \delta\}.$$ Thus we have $\text{length}\left\{ (x, x_{at}) \in (G_d \cap X \leq H)^2 : |t| \ll \delta \right\} \ll \delta \cdot \text{length}(G_d) \ll_{\varepsilon} \delta d^{1+\varepsilon}$ so as $d^{-1/4} \leq \delta$,

$$(\mu_d \times \mu_d)\left\{ (x, x_{at}) \in (G_d \cap X \leq H)^2 : |t| \ll \delta \right\} \ll_{\varepsilon} \delta d^{-1/2} d^{\varepsilon} \ll_{\varepsilon} \delta^3 d^\varepsilon.$$

In the case $q_1 \neq q_2$ Lemma 5.1 and Lemma 5.2 apply and so the length of the set

$$\{(x_1, x_2) \in (G_d \cap X \leq H)^2 : d_X(x_1, x_2) < \delta \text{ and } q_1 \neq q_2\}$$

can be bounded by $\sum_{j=1}^{k} |I_j| \delta \ll \log(d) k \delta \ll_{\varepsilon} H^4 \delta^3 d^{1+2\varepsilon}$. \hfill \Box

References


