On the Construction of Expander Graphs

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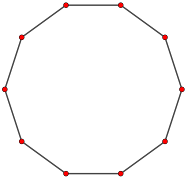
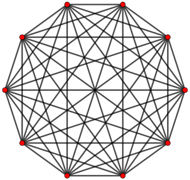
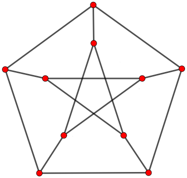
Expander Graphs







Expander Graphs

- $G = (V, E)$ finite graph.
- G is k-regular, i.e. for all $v \in V$,

$$\deg(v) = |\{w \in V \,:\, w \sim v\}| = k.$$

- For $f \in L^2(V)$, define

$$\triangle f(v) = \frac{1}{k} \sum_{w \sim v} f(v) - f(w).$$

- $\triangle$ self-adjoint and has eigenvalues

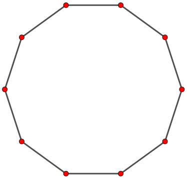
$$\lambda_1 = 0 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq 2.$$

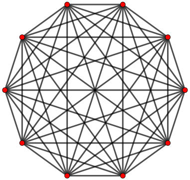
Definition

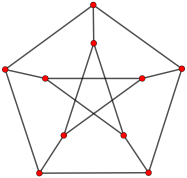
G is ε-**expander** for $\varepsilon > 0$ if

$$\lambda_2(G) \geq \varepsilon.$$

Expander Graphs







$\lambda_2(C_n) \approx \frac{4\pi^2}{n^2}$ $\qquad$ $\lambda_2(K_n) \approx 1$ $\qquad$ $\lambda_2(G) = \frac{2}{3}$

- $\lambda_2(G) > 0$ if and only if G connected.
- Cheeger constant:

$$h(G) = \min_{\substack{A \subset V \\ 0 < |A| \leq \frac{|V|}{2}}} \frac{|E(A, A^c)|}{|A|}.$$

- Cheeger-Buser Inequality:

$$\frac{\lambda_2(G)}{2k} \leq h(G) \leq \sqrt{\frac{2\lambda_2(G)}{k}}.$$

Expander Graphs

If G is a ε-expander, then:

- $$\mathrm{diam}(G) \leq C_\varepsilon \log |G|.$$

- $$||\mu_{v_0}^{(i)} - \frac{1}{|V|}||_2 \leq 2(1-\varepsilon)^i.$$

Expander Graphs

Definition

A sequence of k-regular graphs G_n is an ε-**expander family** if for all n,

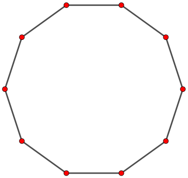
$$\lambda_2(G_n) \geq \varepsilon.$$

Notice G_n has $k|G_n|/2$ many edges.

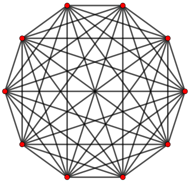
Definition (Cayley Graph)

Let G be a finite group and $S \subset G \backslash \{e\}$ symmetric. Then the **Cayley graph** $\mathrm{Cay}(G, S)$ has vertex set G and

$$E = \{\{g, gs\} : g \in G, s \in S\}.$$



$G = \mathbb{Z}/n\mathbb{Z}, S = \{\pm 1\}.$



$|G| = n, S = G \backslash \{e\}$

Margulis Expander Construction, 1973

Let $S \subset \mathrm{SL}_3(\mathbb{Z})$ be a finite symmetric generating set. Then

$$\mathrm{Cay}(\mathrm{SL}_3(\mathbb{Z}/n\mathbb{Z}), \pi_n(S))$$

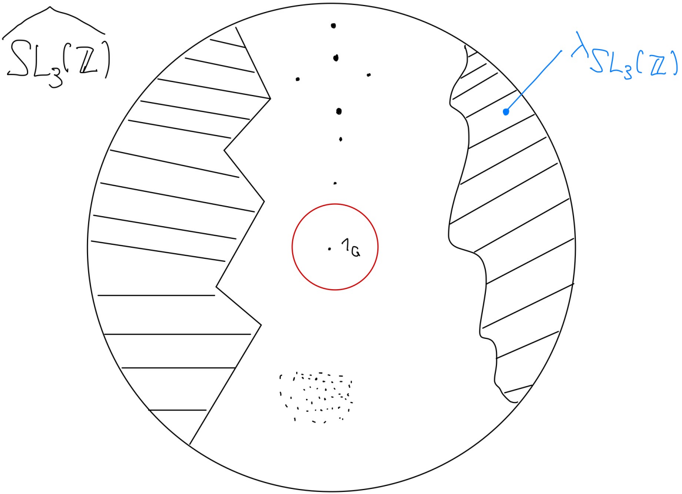
for $n \geq 1$ is an expander family.

Here

$$\pi_n : \mathrm{SL}_3(\mathbb{Z}) \to \mathrm{SL}_3(\mathbb{Z}/n\mathbb{Z}), \qquad g \mapsto (g \bmod n).$$

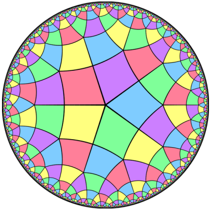
The proof uses that $\mathrm{SL}_3(\mathbb{Z})$ has property (T).

Expander Graphs



Expander Graphs

Denote $\Gamma_n = \{g \in \mathrm{SL}_2(\mathbb{Z}) : g = \mathrm{Id}_2 \mod n\}$. Consider $X_n = \mathbb{H}/\Gamma_n$.



Selberg's Theorem: $\lambda_2(X_p) \geq 3/16$.

Theorem

Let $S \subset \mathrm{SL}_2(\mathbb{Z})$ be a finite symmetric generating set. Then

$$\mathrm{Cay}(\mathrm{SL}_2(\mathbb{F}_p), \pi_p(S))$$

as p ranges among the primes is an expander family.

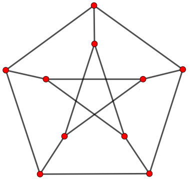
We can bound

$$\lambda_2(G) \leq 1 - \frac{2\sqrt{k-1}}{k} + o_{|G|}(1).$$

Definition

A graph G is called **Ramanujan** if

$$\lambda_2(G) \geq 1 - \frac{2\sqrt{k-1}}{k}$$



Theorem

For any $\gamma > 0$,

$$P\left[\lambda_2(G) \geq \left(1 - \frac{2\sqrt{k-1}}{k} - \gamma\right)\right] \to 1.$$

Theorem (Lubotzky-Phillips-Sarnak, Margulis, 1988)

For any prime p, a family of p-regular Ramanujan graphs exist.

We return to $\mathrm{SL}_n(\mathbb{Z})$. Lubotzky-Problem:

$$S = \left\{ \begin{pmatrix} 1 & \pm 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 3 & 1 \end{pmatrix} \right\}$$

Theorem (Strong Approximation)

Assume $g, h \in \mathrm{SL}_2(\mathbb{Z})$ generate a free group and denote $S = \{g, h, g^{-1}, h^{-1}\}$. Then $\pi_p(S)$ generates $\mathrm{SL}_2(\mathbb{F}_p)$ for almost all primes p.

Do we have expansion?

Theorem (Bourgain-Gamburd, 2008)

Assume $g, h \in \mathrm{SL}_2(\mathbb{Z})$ generate a free group and denote $S = \{g, h, g^{-1}, h^{-1}\}$. Then $\mathrm{Cay}(\mathrm{SL}_2(\mathbb{F}_p), \pi_p(S))$ as p ranges over the primes is an expander family.

Corollary (Bourgain-Gamburd, 2008)

Let $S \subset \mathrm{SL}_2(\mathbb{Z})$ be a finite symmetric subset. Then exactly one of the following holds:

1. $\pi_p(S)$ does not generate $\mathrm{SL}_2(\mathbb{F}_p)$ for large enough primes p.
2. For primes p, $\mathrm{Cay}(\mathrm{SL}_2(\mathbb{F}_p), \pi_p(S))$ is an expander family.

Sum-Product Theorem (Erdös-Szemeredi 1983)

There is $C, \varepsilon > 0$ such that for any finite $A \subset \mathbb{R}$,

$$\max(|A + A|, |A \cdot A|) \geq C|A|^{1+\varepsilon}.$$

Theorem (Bourgain-Katz-Tao, 2003)

For every $\varepsilon > 0$ there is $C(\varepsilon), \delta(\varepsilon) > 0$ such that for any $A \subset \mathbb{F}_p$,

$$\max(|A + A|, |A \cdot A|) \geq C|A|^{1+\varepsilon},$$

provided that $|A| \leq p^{1-\delta}$.

Expansion for $\mathrm{SL}_2(\mathbb{F}_p)$ relies on:

Product Theorem (Helfgott, 2005)

$\exists \varepsilon > 0$ such that for any $A \subset G = \mathrm{SL}_2(\mathbb{F}_p)$ one of the following holds:

(i) (Expansion) $|AAA| \geq |A|^{1+\varepsilon}$.

(ii) (Close to G) $|A| \geq |G|^{1-O(\varepsilon)}$.

(iii) (Trapping) $A \subset H$ for $H \leq G$.

Bourgain-Gamburd furthermore use:

- (Gowers Quasirandomness) A unitary representation of $\mathrm{SL}_2(\mathbb{F}_p)$ has dimension $\geq \frac{p-1}{2}$.
- Non-commutative Balog-Szemeredi-Gowers Lemma: If $A \cdot A$ has a lot of collisions, then there is a close subset $A' \subset A$ such that $|A' \cdot A'| \leq O(|A|)$.

The results of Bourgain-Gamburd had a lot of impact:

1. Generalization to $\mathrm{SL}_n(\mathbb{F}_p)$ and further finite simple groups of Lie type.
2. There is $\varepsilon > 0$ such that the collection of all non-abelian finite simple groups is an expander family (of constant degree).
3. Prime counting
4. Effective equidistribution on compact Lie groups.

Thank you!