The Dimension of Fractal Measures
Results and Open Problems

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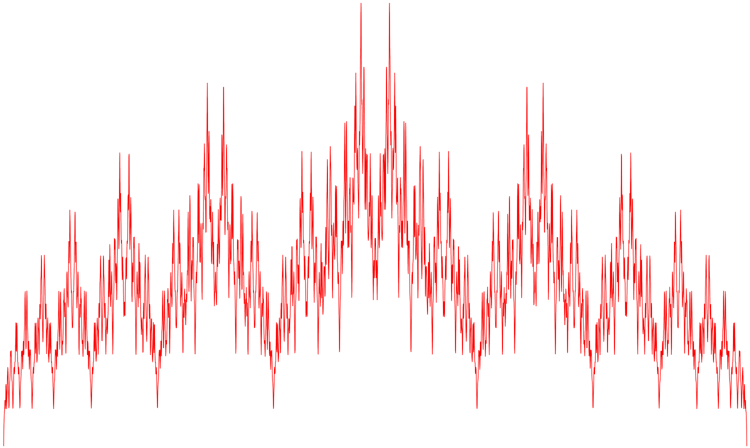
Fractal Measures

$\Leftrightarrow$

Number Theory

Two open problems

1) Bernoulli Convolutions: Random Power Series



Bernoulli Convolutions

- $\lambda \in (0, 1)$
- Consider

$$Y_\lambda = \sum_{i=0}^{\infty} \lambda^i X_i = X_0 + \lambda X_1 + \lambda^2 X_2 + \ldots,$$

  where $X_0, X_1, X_2, \ldots$ are independent random variables satisfying for all $i \geqslant 0$,

$$\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = \frac{1}{2}.$$

- The **Bernoulli convolution** ν_λ of parameter λ is the law of Y_λ, i.e. for $A \subset \mathbb{R}$,

$$\nu_\lambda(A) = \mathbb{P}[Y_\lambda \in A].$$

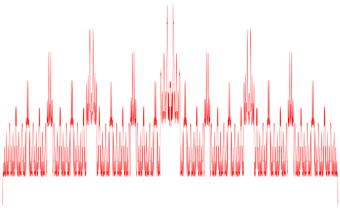
Bernoulli Convolutions

- $\dim \nu_\lambda = \inf\{\dim_H(A) : \nu_\lambda(A) > 0\}$

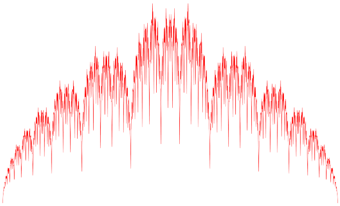
- Exercise: If $\lambda \in (0, \frac{1}{2})$, $\dim \nu_\lambda < 1$.
- Exercise: If $\lambda = \frac{1}{2}$, ν_λ is the normalised Lebesgue measure on $[-2, 2]$ so $\dim \nu_\lambda = 1$.

- Hard: What happens when $\lambda \in (\frac{1}{2}, 1)$?

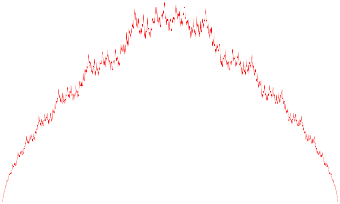
Bernoulli Convolutions



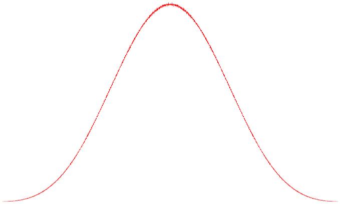
$\lambda = 0.53$



$\lambda = \frac{\sqrt{5}-1}{2} \approx 0.61803$



$\lambda = 0.65$



$\lambda = 0.83$

Bernoulli Convolutions

Theorem (Erdős 1940 (while at IAS), Garsia 1962)

If $\lambda^{-1} \neq 2$ *is a Pisot number, then* $\dim \nu_\lambda < 1$.

- Example: $\frac{\sqrt{5}-1}{2}$. There are infinitely many such λ in $(\frac{1}{2}, 1)$.

Bernoulli Convolutions

Theorem (Hochman 2014 + Varjú 2019)

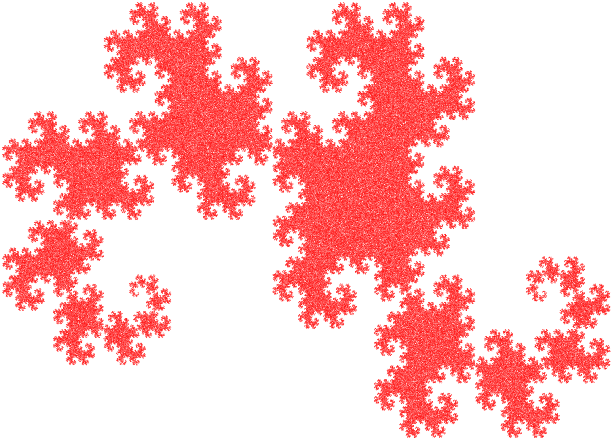
For $\lambda \in (0,1)$*,*

$$\dim \nu_\lambda = \min\left\{1, \frac{h_\lambda}{\log \lambda^{-1}}\right\} \qquad \textit{for} \qquad h_\lambda = \lim_{n\to\infty} \frac{1}{n} H\left(\sum_{i=0}^{n} \lambda^i X_i\right),$$

where $H(\cdot)$ *is the Shannon entropy.*

- If λ is transcendental, $h_\lambda = \log 2$ and so $\dim \nu_\lambda = 1$ if $\lambda \in (\frac{1}{2}, 1)$.
- Open problem: Estimate h_λ to find all algebraic λ with $\dim \nu_\lambda < 1$.

2) Self-similar Measures



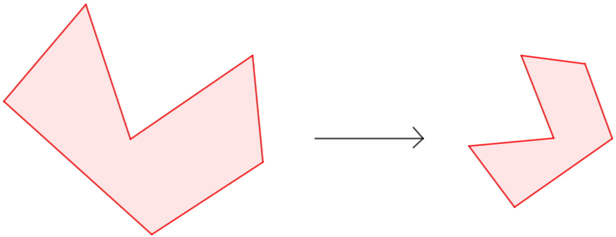
- Let $g_1, \ldots, g_k$ be contractive similarities of $\mathbb{R}^d$, i.e. for $x \in \mathbb{R}^d$,

$$g_i(x) = \rho_i U_i x + b_i$$

with $\rho_i \in (0,1), U_i \in O(d)$ and $b_i \in \mathbb{R}^d$.

- Consider

$$\mu = \sum_{i=1}^{k} p_i \delta_{g_i}.$$

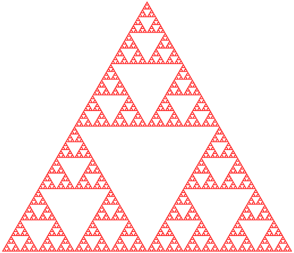


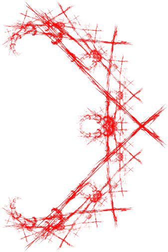
- A **self-similar measure** ν is a stationary measure for

$$\mu = \sum_{i=1}^{k} p_i \delta_{g_i},$$

i.e. the unique probability measure ν satisfying

$$\mu * \nu = \nu.$$





$$\Delta_n = \min\{d(g,h) \,:\, g,h \in \operatorname{supp}(\mu^{*n}) \text{ with } g \neq h\}$$

Theorem (Hochman 2017)

Dimension formula for ν if there is $c > 0$ such that

$$\Delta_n \geqslant e^{-cn} \qquad \textit{for infinitely many } n \geqslant 1. \tag{1}$$

Fact

If $g_i \in G(\overline{\mathbb{Q}})$ there is $c > 0$ such that $\Delta_n \geqslant e^{-cn}$ for all $n \geqslant 1$.

Conjecture

(1) always holds.

Thank you for your attention