JUNIOR NUMBER THEORY SEMINAR (OXFORD)

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I will discuss self-similar measures and touch on their connections to Diophantine approximation and additive combinatorics. Initially, I will address related problems that share a similar general flavor to those explored later. The structure of this talk is as follows:

(1) General Flavor: Diameter on Groups, Additive Combinatorics, and Diophantine Approximation.
(2) Introduction to Self-similar Sets and Self-similar Measures.
(3) Results on Bernoulli Convolutions.
(4) Results on General Self-similar Measures.

1. General Flavor: Diameter on Groups, Additive Combinatorics and Diophantine Approximation

Given a compact metric group G and a subset $S \subset G$, we aim to understand

$$S^n = \{s_1 \cdots s_n \,:\, s_i \in S\}.$$

Additive combinatorics often focuses on the size of S^n. We are additionally interested in the density of S in G. Specifically, we want to estimate the diameter of S in G for a scale $r > 0$ define as

$$\mathrm{diam}_r(G, S) = \min\{n \geq 1 \,:\, S^n \text{ is } r\text{-dense in } G\},$$

where S^n is said to be r-dense in G if for every $g \in G$, $d(g, S^n) < r$.

For instance, consider the case when $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $S_\alpha = \{0, \alpha\}$ for some $\alpha \in \mathbb{R}$. If α is irrational, $S_\alpha^n = \{0, \alpha, 2\alpha, \ldots, n\alpha\}$ and S_α generates a dense subgroup. We seek to understand the diameter. A related question involves examining the difference between two elements of S_α^n. The difference is small whenever

$$n_1\alpha - n_2\alpha \approx m$$

for some integers n_1, n_2 and m. This is equivalent to

$$\alpha \approx \frac{m}{n_1 - n_2},$$

showing that α has a good rational approximation. Consequently, S_α^n becomes denser in $\mathbb{T}$ when α has poor rational approximations, leading us to Diophantine exponents. Notably, Roth proved in 1955 that every algebraic number satisfies

$$\left|q\alpha - p\right| \geq \frac{C(\alpha, \epsilon)}{q^{1+\epsilon}}$$

for every $\epsilon > 0$, with $C(\alpha, \epsilon)$ being a constant depending on α and ϵ. This gives rise to the following corollary.

Corollary 1.1. *For every algebraic number α and $\epsilon > 0$, it holds that*

$$\operatorname{diam}_r(\mathbb{T}, S_\alpha) \ll_{\alpha,\epsilon} r^{-(1+\epsilon)}. \tag{1.1}$$

We also have the following general fact, which relies on the observation that random numbers in $\mathbb{R}$ satisfy Roth's theorem. On the other hand, numbers with very good rational approximations only have bad Diameter estimates.

Corollary 1.2. *The diameter bound* (1.1) *holds for almost every $\alpha \in \mathbb{R}$.*

Corollary 1.3. *Let α be a Liouville number. Then for every n, there exists a scale $r_n \in (0,1)$ such that*

$$\operatorname{diam}_r(\mathbb{T}, S_\alpha) \geq r_n^{-n}.$$

It is intriguing to explore what occurs in other compact groups, such as the special orthogonal group

$$\mathrm{SO}(d) = \{g \in M_n(\mathbb{R}) : gg^T = I \text{ and } \det(g) = 1\},$$

which is a connected compact Lie group. For a subset $S \subset \mathrm{SO}(d)$, we similarly aim to study the diameter $\operatorname{diam}_r(\mathrm{SO}(d), S)$. Here, $|S^n|$ generally grows exponentially, as $\mathrm{SO}(d)$ is non-abelian. We have the following significant result, which can be viewed as an analogue of Roth's theorem.

Theorem 1.4. *(Bourgain-Gamburd 2008, 2012; Benoist-de Saxcé 2016) Let $d \geq 3$ and $S \subset \mathrm{SO}(d)$ be a finite subset generating a dense subgroup, assuming all elements of S have algebraic entries. Then*

$$\operatorname{diam}(\mathrm{SO}(d), S) \ll_S \log(r^{-1}). \tag{1.2}$$

The proof of this theorem relies on additive combinatorics (discretized product theorems on Lie groups) and representation theory. Indeed, one first shows that $d(S^n, e) \geq e^{-cn}$ for some $c > 0$ and all $n \geq 1$ (or rather a weakening of that) and then deduces the claim under this assumption. It relates to similar results for Cayley graphs of groups of Lie type (as seen in works by Helfgott, Bourgain-Gamburd, Pyber-Szabo, and Breuillard-Green-Tao). Moreover, something stronger holds: if $\mu_S = \frac{1}{|S|}\sum_{s\in S}\delta_s$, then $\mu_S^{*n} \to \mathrm{vol}_{\mathrm{SO}(d)}$ as $n \to \infty$, where $\mathrm{vol}_{\mathrm{SO}(d)}$ denotes the volume probability measure (Haar measure) on $\mathrm{SO}(d)$. However, the following problems remain open.

Conjecture 1.5. *(Sarnak's Conjecture) The diameter bound* (1.2) *holds for $S = \{a, b\}$ and almost all $a, b \in \mathrm{SO}(d)$ for $d \geq 3$.*

Conjecture 1.6. *The diameter bound* (1.2) *holds for every $S \subset \mathrm{SO}(d)$ generating a dense subgroup.*

Both of these conjectures conjecture are wide open, and we have no examples beyond the algebraic case.

2. Self-similar sets and self-similar measures

Recall that if $g : \mathbb{R}^d \to \mathbb{R}^d$, then

$$\operatorname{Lip}(g) = \sup\left\{\frac{|g(x) - g(y)|}{|x - y|} : x, y \in \mathbb{R}^d \text{ with } x \neq y\right\}.$$

Theorem 2.1. *(Hutchinson 1981) Let* $g_1, \ldots, g_n : \mathbb{R}^d \to \mathbb{R}^d$ *be maps with* $\mathrm{Lip}(g_i) < 1$ *for all* $1 \leq i \leq n$*. Then there exists a unique compact set* $K \subset \mathbb{R}^d$ *such that*

$$K = \bigcup_{i=1}^{n} g_i(K).$$

Proof. (Sketch) By Banach's Fixed Point Theorem, for every $g : \mathbb{R}^d \to \mathbb{R}^d$ with $\mathrm{Lip}(g) < 1$, there is a unique fixed point $z = \mathrm{Fix}(g) \in \mathbb{R}^d$ (i.e., satisfying $g(z) = z$). Consider the map

$$\Phi : \{1, \ldots, n\}^{\mathbb{N}} \to \mathbb{R}^d, \quad (i_1, i_2, \ldots) \mapsto \lim_{\ell \to \infty} \mathrm{Fix}(g_{i_1} \circ \ldots \circ g_{i_\ell})$$

and set $K = \mathrm{Im}(\Phi)$.

To show uniqueness, assume such a set K exists. Then it follows that

$$K = \bigcup_{i_1, \ldots, i_\ell = 1}^{n} (g_{i_1} \circ \ldots \circ g_{i_\ell})(K).$$

One can verify that for every sequence $(i_1, i_2, \ldots)$, it holds that

$$\bigcap_{\ell \geq 1} (g_{i_1} \circ \ldots \circ g_{i_\ell})(K) = \left\{ \lim_{\ell \to \infty} \mathrm{Fix}(g_{i_1} \circ \ldots \circ g_{i_\ell}) \right\}.$$

□

The set K is called an attractor. We want to consider a special class of attractors, namely self-similar sets which arise from similarities. Specifically, we consider the maps $g_1, \ldots, g_n$ to be similarities, that is, maps for which there exists $\rho > 0$ such that $d(g(x), g(y)) = \rho d(x, y)$ for all $x, y \in \mathbb{R}^d$. Every similarity $g : \mathbb{R}^d \to \mathbb{R}^d$ can be expressed as

$$g(x) = \rho(g) U(g) x + b(g)$$

for a scalar $\rho(g) \in \mathbb{R}_{>0}$, an orthogonal matrix $U(g) \in O(d)$, and a vector $b(g) \in \mathbb{R}^d$.

To provide a few examples:

(1) If there exists $x \in \mathbb{R}^d$ such that $g_i(x) = x$ for all $1 \leq i \leq n$, then $K = \{x\}$.

(2) (Cantor Set) If $d = 1$ and $\lambda \in (0, 1/2)$, then consider

$$g_1(x) = \lambda x \quad \text{and} \quad g_2(x) = \lambda x + (1 - \lambda). \tag{2.1}$$

The resulting self-similar set $\mathcal{C}_\lambda$ is known as a Cantor set.

Conversely, if we consider the same maps with $\lambda \in (1/2, 1)$, then $K = [0, 1]$.

(3) (Sierpinski Triangle) Let $d = 2$ and consider the Sierpinski triangle arising from the similarities

$$g_1(x) = \frac{1}{2}x, \quad g_2(x) = \frac{1}{2}x + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \quad \text{and} \quad g_3(x) = \frac{1}{2}x + \begin{pmatrix} 1/4 \\ \sqrt{3}/4 \end{pmatrix}.$$

The main question in the study of self-similar sets is to determine their dimension. It is usually essential to specify the notion of dimension being used, but it is a classical fact that, for self-similar sets, all major notions of dimension coincide, particularly the Hausdorff dimension and the box (Minkowski) dimension. We now define the box dimension. For a subset $Y \subset \mathbb{R}^d$, denote its covering number at scale $r > 0$ by

$$N_r(Y) = \min\{k : Y \text{ can be covered by } k \text{ sets of diameter} \leq r\}.$$

The **box dimension** of Y, if it exists, is the exponential growth rate of $N_r(Y)$:

$$\dim Y = \lim_{r\to 0} \frac{\log N_r(Y)}{\log(1/r)}.$$

Equivalently, $\dim Y = \alpha$ if and only if $N_r(Y) = r^{-\alpha+o(1)}$ as $r \to 0$. Calculating the dimension is generally challenging, yet in the previously discussed cases, it is straightforward. We note that the Cantor set satisfies $\dim(\mathcal{C}_\lambda) = \frac{\log(2)}{\log(\lambda^{-1})}$, while the Sierpinski triangle has dimension $\frac{\log(3)}{\log(2)}$. Recently, there has been significant progress in studying the dimension of self-similar sets. We state a special case of Hochman's recent work.

Theorem 2.2. *(Hochman 2014, special case) Let $g_1, \dots, g_n$ be similarities on $\mathbb{R}^d$ satisfying the following properties:*

(1) There exists $\rho \in (0,1)$ such that $\rho = \rho(g_i)$ for all $1 \le i \le n$.
(2) The $g_1, \dots, g_n$ generate a free semi-group.
(3) The coefficients of g_i for all $1 \le i \le n$ are algebraic.

Then for the self-similar set K, it holds that

$$\dim K = \min\left\{d, \frac{\log n}{\log(1/\rho)}\right\}.$$

Instead of studying self-similar sets, we can study measures supported on self-similar sets. This leads to the notion of self-similar measures, which form a richer class of objects to study. Hutchinson's theorem can be generalized in this context.

Theorem 2.3. *(Hutchinson 1981) Let μ be a probability measure supported on finitely may similarities $g_1, \dots, g_n$ satisfying $\rho(i) < 1$ for all $1 \le i \le n$. Then there exists a unique compactly supported probability measure ν on $\mathbb{R}^d$ that is μ-stationary, i.e.,*

$$\mu * \nu = \nu.$$

The measure ν is called the self-similar measure of μ. It is well-known that ν is exact dimensional, meaning there exists some $\alpha \in [0, d]$, called the dimension of ν, such that for ν-almost all $x \in \mathbb{R}^d$ we have

$$\nu(B_r(x)) = r^{\alpha + o_{\nu,x}(1)}.$$

One is typically interested in the following questions:

(1) What is $\dim \nu$?
(2) Is ν absolutely continuous? That is, is there $f_\nu \in L^1(\mathbb{R}^d)$ such that $\nu = f_\nu \cdot \mathrm{vol}_{\mathbb{R}^d}$?

We note that if ν is absolutely continuous, then $\dim \nu = 1$. We will first review a specific class of self-similar measures.

3. Bernoulli Convolutions

A particularly well-studied of self-similar measures are Bernoulli convolutions, which arise from the maps (2.1). Specifically, for $\lambda \in (0,1)$, we denote by ν_λ the self-similar measure of

$$\frac{1}{2}\delta_{g_1} + \frac{1}{2}\delta_{g_2}.$$

The measure ν_λ is referred to as the Bernoulli convolution of λ. An alternative description is that ν_λ has, up to translation and rescaling, the same distribution as

$\sum_{i=0}^{\infty} X_i \lambda^i$, where X_i are independent random variables satisfying $\mathbb{P}[X_i = -1] = \mathbb{P}[X_i = 1] = \frac{1}{2}$. We make the following basic observations:

(1) When $\lambda \in (0, 1/2)$, then $\mathrm{supp}(\nu_\lambda) = C_\lambda$ and ν_λ is not absolutely continuous.
(2) For $\lambda = 1/2$, ν_λ is the Lebesgue measure on $[0, 1]$.
(3) For $\lambda \in (1/2, 1)$, $\mathrm{supp}(\nu_\lambda) = [0, 1]$ and ν_λ may be absolutely continuous or not.

Here is a selection of known results on Bernoulli convolutions in their historical order:

(1) (Erdös 1939, Garcia 1962) When λ^{-1} is a Pisot number, then $\dim(\nu_\lambda) < 1$ and so ν_λ is not absolutely continuous. (A Pisot number is one where all of its Galois conjugates have modulus < 1, for example $\lambda = \frac{\sqrt{5}-1}{2}$.) This uses that Pisot numbers are very closely approximated by integers.
(2) (Erdös 1940) There exists some $c \geq 1/2$ such that ν_λ is almost surely absolutely continuous in $[c, 1]$. Erdös conjectured that one can take $c = 1/2$.

Next, we recall the definition of the Mahler measure M_λ. We recall if $H(\lambda)$ is the algebraic height of λ, then $H(\lambda) = M_\lambda^{1/d}$ for d the degree of the algebraic number.

Definition 3.1. *The Mahler measure M_λ of an algebraic number λ is defined as*

$$M_\lambda = |a| \prod_{|z_j|>1} |z_j|$$

where $a(x - z_1) \cdots (x - z_\ell)$ is the minimal polynomial of λ over $\mathbb{Z}$.

(3) (Garcia 1962) ν_λ is absolutely continuous if λ^{-1} is an algebraic integer and $M_\lambda = 2$. Examples include $\lambda = 2^{-1/k}$ and the real roots of $x^{p+n} - x^n - 2$ for any $\max\{p, n\} \geq 2$.
(4) (Solomyak 1995) ν_λ is absolutely continuous for almost all $\lambda \in [1/2, 1]$.
(5) (Hochman 2014) Provided a dimension formula for ν_λ when λ is algebraic and also showed that if $\dim \nu_\lambda < 1$, then λ satisfies a Diophantine approximation condition.
(6) (Varjú 2019) Showed that there exists a small $c > 0$ such that when λ is algebraic and satisfies

$$\lambda > 1 - c \min\{\log M_\lambda, (\log M_\lambda)^{-2}\}$$

then ν_λ is absolutely continuous. For instance, this implies that ν_λ is absolutely continuous for $\lambda = 1 - \frac{1}{n}$ for all $n \geq 10^{50}$. This result was strengthened by Kittle-Kogler in 2024.
(7) (Varjú 2019) Established that $\dim \nu_\lambda = 1$ for transcendental $\lambda \in (1/2, 1)$.
(8) (Klepsyn-Pollicott-Vytnova 2022) Demonstrated that $\dim \nu_\lambda \geq 0.96399$ for all $\lambda \in (1/2, 1)$.

4. Results on General Self-Similar Measures

The dimension theory of a general self-similar measure ν is quite well understood. For simplicity of this exposition we focus on the case when $d = 1$. Specifically, we denote by

$$h_\mu = \lim_{n\to\infty} \frac{1}{n} H(\mu^{*n}) = \inf_{n\geq 1} \frac{1}{n} H(\mu^{*n})$$

the random walk entropy of μ, and by χ_μ the Lyapunov exponent defined as

$$\chi_\mu = \mathbb{E}_{g\sim\mu}[\log \rho(g)].$$

It is well-known that

$$\dim \nu \leq \min\left\{1, \frac{h_\mu}{|\chi_\mu|}\right\}$$

and it is conjectured that this inequality is always an equality.

Theorem 4.1. *(Hochman 2014) Let μ be a probability measure supported on finitely may similarities $g_1, \dots, g_n$ on $\mathbb{R}$ satisfying $\rho(g_i) < 1$ for all $1 \leq i \leq n$. Assume that all $g \in \operatorname{supp}(\mu)$ have algebraic coefficients. Then*

$$\dim \nu = \min\left\{1, \frac{h_\mu}{|\chi_\mu|}\right\}. \tag{4.1}$$

In light of this result, it is conjectured that if

$$\frac{h_\mu}{|\chi_\mu|} > 1,$$

then ν is absolutely continuous. This conjecture remains wide open, also for Bernoulli convolutions, where Varjú's result are progress towards it. Together with Samuel Kittle, I have made some advances on this conjecture. A simplified version of what we have proved is the following.

Theorem 4.2. *(Kittle-Kogler 2024, Special case) For every $\varepsilon > 0$ there exists a constant $C > 1$ and $\hat{\rho} \in (0,1)$ such that the following holds. Let $\mu = \sum_{i=1}^n \delta_i g_i$ be a probability measure supported on finitely may similarities $g_1, \dots, g_n$ on $\mathbb{R}$ with $\delta_i \geq \varepsilon$ and $\rho(g)_i \in (\hat{\rho}, 1)$ for all $1 \leq i \leq n$. Assume furthermore that all of the coefficients of $g_1, \dots, g_n$ are in a number field K and have height at most H. Then ν is absolutely continuous if*

$$\frac{h_\mu}{|\chi_\mu|} \geq C[K : \mathbb{Q}] \max\left\{1, \log\left(\frac{\log H}{h_\mu}\right)\right\}^2.$$