On Dimension and Absolute Continuity of Self-Similar Measures

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Plan for talk:

1. Introduction
2. Dimension Results
3. Results on Absolute Continuity

Introduction

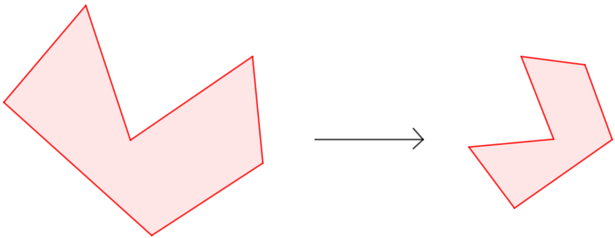
- A **similarity** is a map $g : \mathbb{R}^d \to \mathbb{R}^d$ such that $\exists \rho > 0$ satisfying

$$||g(x) - g(y)||_2 = \rho \cdot ||x - y||_2$$

  for all $x, y \in \mathbb{R}^d$.
- Every similarity $g : \mathbb{R}^d \to \mathbb{R}^d$ can be written as

$$g(x) = \rho U x + b$$

  for all $x \in \mathbb{R}^d$ with $\rho \in \mathbb{R}_{>0}, U \in O(d)$ and $b \in \mathbb{R}^d$.



Introduction

- Fix a probability measure supported on finitely many similarities

$$\mu = \sum_{i=1}^{\ell} p_i \delta_{g_i} \qquad \text{for} \qquad g_i(x) = \rho_i U_i x + b_i$$

  with $\rho_i \in \mathbb{R}_{>0}$, $U_i \in O(d)$ and $b_i \in \mathbb{R}^d$.
- The **Lyapunov exponent** of μ is defined as

$$\chi_\mu = \mathbb{E}_{g\sim\mu}[\log \rho(g)] = \sum_{i=1}^{\ell} p_i \log \rho_i.$$

- μ **contracting on average** if $\chi_\mu < 0$.
- μ **contracting** if $\rho_i \in (0, 1)$ for all i

Introduction

- If $\chi_\mu < 0$, then there exists a probability measure ν on $\mathbb{R}^d$ such that

$$\mu^{*n} * \delta_{x_0} \to \nu$$

for all $x_0 \in \mathbb{R}^d$.

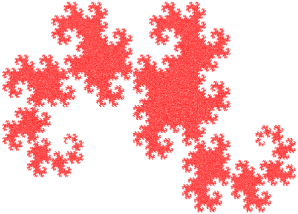
- ν is called the **self-similar measure** of μ.
- ν is also the unique μ-stationary probability measure on $\mathbb{R}^d$, that is

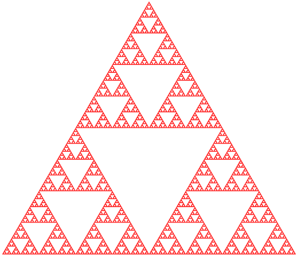
$$\mu * \nu = \sum_{i=1}^{\ell} p_i g_i \nu = \nu.$$

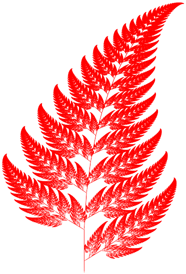
- When μ is contracting, $K = \mathrm{supp}(\nu)$ is compact and it holds that

$$K = \bigcup_{g \in \mathrm{supp}(\mu)} g(K). \qquad (1)$$

Images of Self-Similar Sets







Introduction

Main questions in the area:

- What is the $\dim \nu$?
  It is known by Feng-Hu 2009 and Feng 2023 that

  $$\nu(B_r(x)) = r^{\dim \nu + o_{\nu,x}(1)}$$

  for ν-almost all $x \in \mathbb{R}^d$.
- Is ν absolutely continuous, i.e. is there $f \in L^1(\mathbb{R}^d)$ such that

  $$d\nu = f \, d\mathrm{vol}_{\mathbb{R}^d}$$

Dimension Results

- The **random walk entropy** is defined as

$$h_\mu = \lim_{n\to\infty} \frac{1}{n} H(\mu^{*n}) = \inf_{n\geqslant 1} \frac{1}{n} H(\mu^{*n}),$$

  where $H(\cdot)$ is the Shannon entropy.
- When the support of μ generates a free semi-group, then

$$h_\mu = H(\mu) = -\sum_{i=1}^{\ell} p_i \log p_i.$$

- Fact:

$$\dim \nu \leqslant \min\left\{d, \frac{h_\mu}{|\chi_\mu|}\right\}.$$

Dimension Results

Recall that $\rho_i \in \mathbb{R}_{>0}, U_i \in O(d)$ and $b_i \in \mathbb{R}^d$ and

$$\mu = \sum_{i=1}^{\ell} p_i \delta_{g_i} \qquad \text{with} \qquad g_i(x) = \rho_i U_i x + b_i.$$

For the remainder of this talk assume:

1. $U_1, \ldots, U_\ell$ acts irreducibly on $\mathbb{R}^d$, i.e. there are no invariant subspaces other than $\{0\}$ and $\mathbb{R}^d$.
2. The g_i do not have a common fixed point.

Conjecture (Generalised exact overlaps conjecture)

Under these assumptions,

$$\dim \nu = \min\left\{d, \frac{h_\mu}{|\chi_\mu|}\right\}. \tag{2}$$

Dimension Results

- Write

$$M_n = \min\left\{ d(g,h) \ : \ \text{for } g,h \in \bigcup_{i=1}^{n} \mathrm{supp}(\mu^{*i}) \text{ with } g \neq h \right\}.$$

  for d a natural metric on the group of similarities.
- Fact: If ρ_i, U_i and b_i all have algebraic coefficients, then for some $c > 0$ we have

$$M_n \geqslant e^{-cn} \qquad \text{for all} \qquad n \geqslant 1$$

Dimension Results

Theorem (Hochman 2014, 2017)

Assume μ is contracting. Then $\dim \nu = \min\left\{d, \frac{h_\mu}{|\chi_\mu|}\right\}$ holds if there exists $c > 0$ such that

$$M_n \geqslant e^{-cn}$$

for infinitely many n.

Theorem (Kittle-Kogler 2025)

Assume $\chi_\mu < 0$. Then $\dim \nu = \min\left\{d, \frac{h_\mu}{|\chi_\mu|}\right\}$ holds if either:

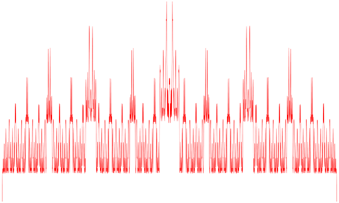
1. *There exists $c > 0$ such that $M_n \geqslant e^{-cn}$ for infinitely many n.*
2. *For all sufficiently large n,*

$$\log M_n \geqslant -n \exp(\log(n)^{1/4}).$$

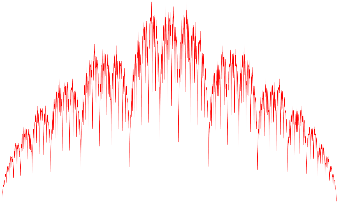
Application: Complex Bernoulli Convolutions

- Let $\lambda \in \mathbb{C}$ with $|\lambda| < 1$.
- Consider
$$Y_\lambda = \sum_{i=0}^{\infty} \lambda^i X_i = X_0 + \lambda X_1 + \lambda^2 X_2 + \dots,$$
where $X_0, X_1, X_2, \dots$ are independent random variables satisfying for all $i \geqslant 0$,
$$\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = \frac{1}{2}.$$
- The **Bernoulli convolution** ν_λ of parameter λ is the law of Y_λ, i.e. for $A \subset \mathbb{C}$,
$$\nu_\lambda(A) = \mathbb{P}[Y_\lambda \in A].$$
- When $\lambda \in \mathbb{R}$, then ν_λ is a probability measure on $\mathbb{R}$.

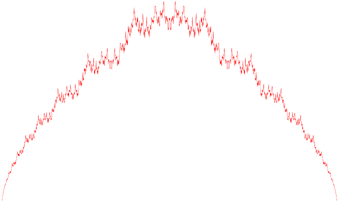
Images of Real Bernoulli Convolutions



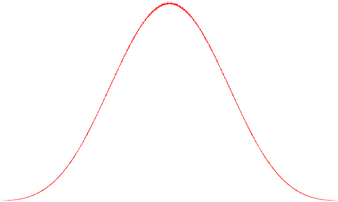
$\lambda = 0.53$



$\lambda = \frac{\sqrt{5}-1}{2} \approx 0.61803$



$\lambda = 0.65$



$\lambda = 0.83$

Application: Complex Bernoulli Convolutions

Theorem (Hochman 2014 + Varjú 2019)

The generalised exact overlaps conjecture holds for real Bernoulli convolutions.

Theorem (Kittle-Kogler 2026)

The generalised exact overlaps conjecture holds for complex Bernoulli convolutions.
In particular, if $\lambda \in \mathbb{C}\backslash\mathbb{R}$ is a transcendental number with $|\lambda| < 1$. Then

$$\dim \nu_\lambda = \min\left\{2, \frac{\log 2}{\log |\lambda|^{-1}}\right\}.$$

In particular, $\dim \nu_\lambda = 2$ *when additionally* $|\lambda| \in [2^{-1/2}, 1)$*.*

Results on Absolute Continuity

Conjecture

ν is absolutely continuous if $\frac{h_\mu}{|\chi_\mu|} > d$.

- Write $S_n = -\frac{1}{n} \log M_n$ and $S_\mu = \limsup_{n\to\infty} S_n$.

Theorem (Kittle-Kogler 2024)

Fix $(U_1, \ldots, U_\ell)$ and $(p_1, \ldots, p_\ell)$ and assume $\rho_i \in (0,1)$ for all i. Then there exists a constant C such that ν is absolutely continuous if

$$\frac{h_\mu}{|\chi_\mu|} > C\left(\max\left\{1, \log\frac{S_\mu}{h_\mu}\right\}\right)^2.$$

- Fact: If all of the coefficients of ρ_i, U_i, b_i lie in a number field K and have logarithmic height $\leqslant L$, then $S_\mu \ll_d L \cdot [K : \mathbb{Q}]$.
- C can be made uniform for perturbations of the U_i.

Results on Absolute Continuity

- Corollary: Let $d \geqslant 1$ and fix U_i and b_i with algebraic entries. Then ν is absolutely continuous if for sufficiently large primes q and

$$g_i(x) = \frac{q}{q+i} U_i x + b_i.$$

- We get a similar result when $\chi_\mu < 0$ under some additional assumption. For example, we can show that in $d = 1$ that when

$$g_1(x) = \frac{n}{n+1} x \qquad \text{and} \qquad g_2(x) = \frac{n+1}{n} x + 1,$$

then the self-similar measure of $\frac{4}{5}\delta_{g_1} + \frac{1}{5}\delta_{g_2}$ is absolutely continuous for sufficiently large $n \geqslant 1$.

- Corollary: Strengthening of Varjú's result for Bernoulli convolutions. For example, there is an absolute constant $c > 0$ such that ν is absolutely continuous if

$$\lambda = 1 - \frac{p}{q} \qquad \text{and} \qquad p \leqslant c \frac{q}{(\log \log q)^2}.$$

- Analogous result for complex Bernoulli convolutions.

Results on Absolute Continuity

Our theorem also implies the following version of the main result of Lindenstrauss-Varjú:

Theorem (Lindenstrauss-Varjú 2016)

Let $d \geqslant 3$ and fix $(U_1, \ldots, U_\ell)$ and $(p_1, \ldots, p_\ell)$. Assume that $U_1, \ldots, U_\ell$ generates a dense subgroup of $\mathrm{SO}(d)$ and all entries are algebraic. Then there exists $\tilde{\rho} \in (0,1)$ depending on U_i and p_i such that ν is absolutely continuous if

$$\rho_i \in (\tilde{\rho}, 1) \qquad \textit{for all} \qquad 1 \leqslant i \leqslant \ell.$$

- In Lindenstrauss-Varjú, $\tilde{\rho}$ depends on the $L^2(\mathrm{SO}(d))$ spectral gap of the U_i.
- Our methods are softer: We get uniformity on perturbations of the U_i and do not require that the U_i generate a dense subgroup, but only a non virtually solvable one.

Some comments on the proofs

Two methods, first introduced by Kittle in his work on absolutely continuous Furstenberg measures:

1. Variance Summation Method: Decomposing

$$\mu^{*n} * \delta_{x_0} \qquad \text{into} \qquad x_0 + X_1 \ldots + X_n$$

   and summing up variance contributions at various scales.
2. Detail of a Measure $s_r(\nu)$ (introduced by Kittle in 2021):
   2.1 Tool to prove absolute continuity.
   2.2 Behaves well under convolutions of measures.

Thank you for your attention