Stationary Measures for Random Walks on Lie Groups

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4th July 2025

Goal of Talk

- Kyoshi Ito's first paper.
- Show pictures of random walks on Lie groups.

Context

- Markov chains on finite state spaces have often stationary distributions.
- SDEs too.
  For example the Ornstein-Uhlenbeck process

$$dX_t = -\theta X_t dt + \sigma dW_t$$

  has stationary distribution $\mathcal{N}(0, \sigma^2/2\theta)$.
- But not always! $dX_t = dW_t$ is a pure Brownian motion.

Random Walks on Groups

- Consider group action $G \curvearrowright X$.
- Let μ be a probability measure on G and let

$$Z_1, Z_2, Z_3, \ldots$$

  be independent μ-distributed random variables on G.
- For $x_0 \in X$ consider

$$Y_{n,x_0} = Z_1 \cdots Z_n . x_0$$

- We aim to understand

$$\mathbb{P}[Y_{n,x_0} \in B] = (\mu^{*n} * \delta_{x_0})(B)$$

  for subsets $B \subset X$.
- Today: G Lie Group and μ finite support.

1) Random Walks on Compact Groups

Ito's first paper

- G compact topological group such as

$$\text{finite groups}, \quad (\mathbb{R}/\mathbb{Z})^d, \quad \mathrm{SO}(d), \quad \mathbb{Z}_p, \quad \mathrm{SL}_n(\mathbb{Z}_p), \quad \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

- Then $\exists!$ Haar probability measure vol_G, i.e.

$$g_* \mathrm{vol}_G = \mathrm{vol}_G$$

  for all $g \in G$.
- Assume μ **aperiodic** probability measure on G, i.e. $\mathrm{supp}(\mu) \not\subset gH$ for $g \in G$ and $H \subset G$ a proper closed subgroup of G.
- **Theorem (Ito-Kawada 1940)** Then the Haar measure vol_G is the unique μ-stationary measure, i.e. satisfying $\mu * \mathrm{vol}_G = \mathrm{vol}_G$. Moreover,

$$\mu^{*n} \to \mathrm{vol}_G$$

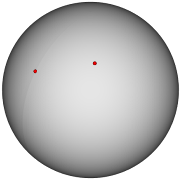
  as $n \to \infty$ in the vague topology.

Random Walks on Compact Lie Groups

- G compact simple Lie group with Haar probability measure vol_G.
- Example: $\mathrm{SO}(3) = \{g \in \mathrm{M}_3(\mathbb{R}) \, : \, g^T = g^{-1} \text{ \& } \det(g) = 1\}$.
- Conjecture: If $\mathrm{supp}(\mu)$ generates a dense subgroup then

$$\mu^{*n}(B) = \mathrm{vol}_G(B) + O_{\mu,B}(e^{-cn}).$$

- **Theorem (Bourgain-Gamburd 2008, Benoist-de Saxce 2015)** Conjecture holds if $\mathrm{supp}(\mu)$ is in addition supported on matrices with **algebraic entries**.



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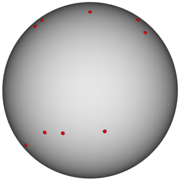

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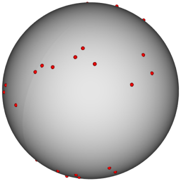

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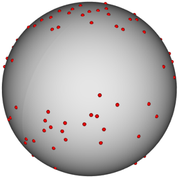

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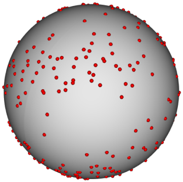

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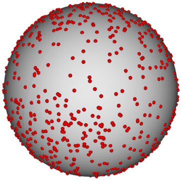
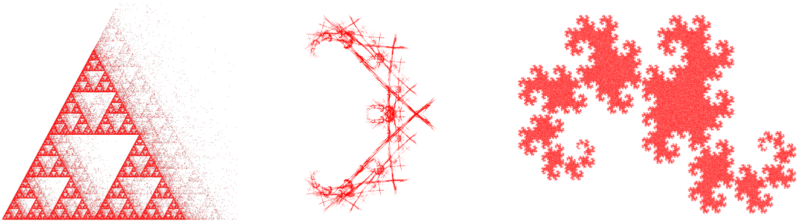


2) Self-Affine Measures



Self-Affine Measures

- Consider **contracting** affine maps, that is maps $g_i : \mathbb{R}^d \to \mathbb{R}^d$ for $1 \leqslant i \leqslant \ell$ given for $x \in \mathbb{R}^d$ by

$$g_i(x) = A_i x + b_i$$

  with $A_i \in M_n(\mathbb{R}^d)$, $b_i \in \mathbb{R}^d$ and $||A_i|| < 1$.
- Let

$$\mu = \sum_{i=1}^{\ell} p_i \delta_{g_i},$$

  which is a probability measure on the **Lie group of affine maps**.
- Then there exists a unique measure ν on $\mathbb{R}^d$, called the **self-affine measure** such that on $\mathbb{R}^d$

$$\mu * \nu = \nu.$$

- Moreover, for any $x_0 \in \mathbb{R}^d$,

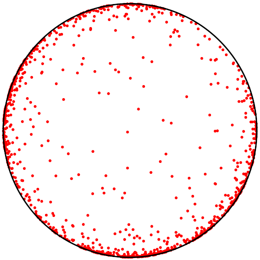
$$\mu^{*n} * \delta_{x_0} \to \nu.$$

Self-Affine Measures

1. What is $\dim \nu$?
2. Is ν absolutely continuous, i.e. is there $f \in L^1(\mathbb{R}^d)$ such that

$$\nu = f \cdot d\mathrm{vol}_{\mathbb{R}^d}.$$

3) Local Limit Theorem on Symmetric Spaces

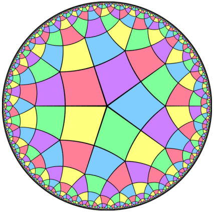
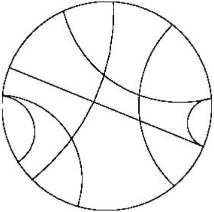


Local Limit Theorem on Symmetric Spaces

- Context: Local limit theorem on $\mathbb{R}$. If $\mathrm{Law}(Z_i)$ is non-lattice (i.e. not supported on $\alpha\mathbb{Z}$ for $\alpha \in \mathbb{R}$) and centered. Then $Y_n = Z_1 + \ldots + Z_n$ satisfies

$$\lim_{n\to\infty} \sqrt{n} \cdot \mathbb{P}[Y_n \in [a, b]] = c \cdot (b - a)$$

  for $c > 0$ a constant depending only on $\mathrm{Law}(Z_i)$.
- $\mathrm{SL}_2(\mathbb{R}) = \{g \in M_2(\mathbb{R}) : \det(g) = 1\}$.
  $\mathrm{SL}_2(\mathbb{R})$ is the isometry group of hyperbolic space.
- Open Problem: Does a local limit theorem hold for $\mathrm{SL}_2(\mathbb{R})$?

Local Limit Theorem on Symmetric Spaces

- In some cases a local limit theorem is known:
  (Bougerol 1982, Kogler 2022)
- If $\mathrm{supp}(\mu)$ is a probability measure on $\mathrm{SL}_2(\mathbb{R})$ that satisfies some assumptions then

$$\lim_{n \to \infty} \frac{n^{3/2}}{\sigma^n} \mu^{*n}(B) = \nu_\mu(B)$$

  for $\sigma = ||\lambda_G(\mu)||_{\mathrm{op}} < 1$ and ν_μ a measure on G.
- The measure ν_μ is μ-stationary ($\mu * \nu_\mu = \nu_\mu$) and not the Haar measure.

Thank you for your attention