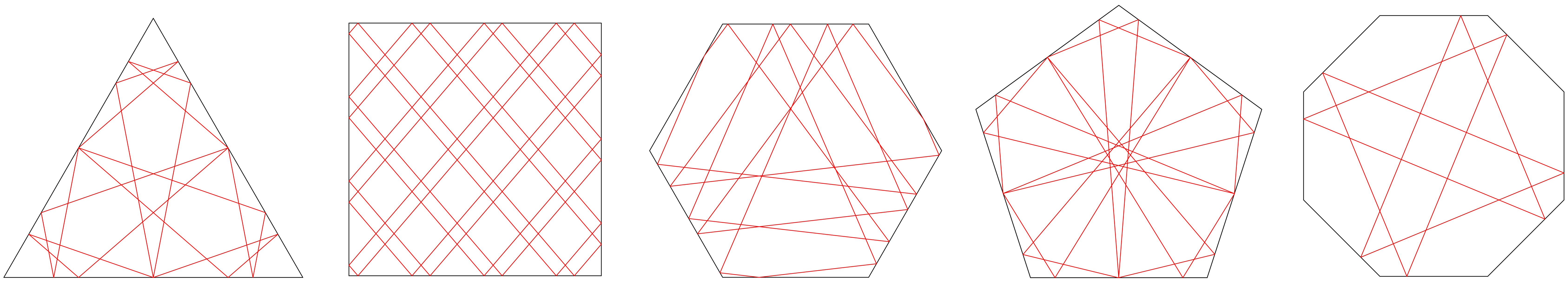
Constantin Kogler, Switzerland

On periodic billiard trajectories in regular polygons and simple closed geodesics on the tetrahedron, cube and octahedron



Introduction

What is mathematical billiard?

Mathematical billiard is about the frictionless motion of a mass point, called billiard ball, in a bounded domain. The billiard ball moves along a straight line with a constant speed until it hits the boundary. The reflection off the boundary is elastic, which means the angle of incidence is equal to the angle of reflection. After the reflection, the point continues its motion until it hits the boundary again. There are two types of billiard trajectories:

(1) periodic billiard trajectory: repeats itself (as you can see in the pictures above)
(2) non-periodic billiard trajectory: does not repeat itself

In general polygons it is difficult to classify periodic billiard trajectories. However, it gets simpler if we examine regular polygons.

What is a platonic solid?

A platonic solid is a regular polyhedron. It consists of regular polygons and the same number of faces meet at each vertex. There exist exactly five and they are called: Tetrahedron, Cube, Octahedron, Icosahedron and Dodecahedron.

The paper examined:

(1) Periodic billiard trajectories in regular polygons
(2) Periodic billiard trajectories in a cube
(3) Geodesics (straight curves) on platonic solids

Methods and Results

Periodic billiard trajectories in regular polygons

Without loss of generality, we say the starting point is on one of the edges and all edges have a length of 1. Let's investigate a special depiction of the billiard trajectory, the so called billiard trajectory in the net. The net is constructed as follows: For each reflection, instead of reflecting the billiard ball on the edge, we reflect the polygon on the edge. Thereby the billiard trajectory is transformed into a straight line - as you can see in Figure 1.

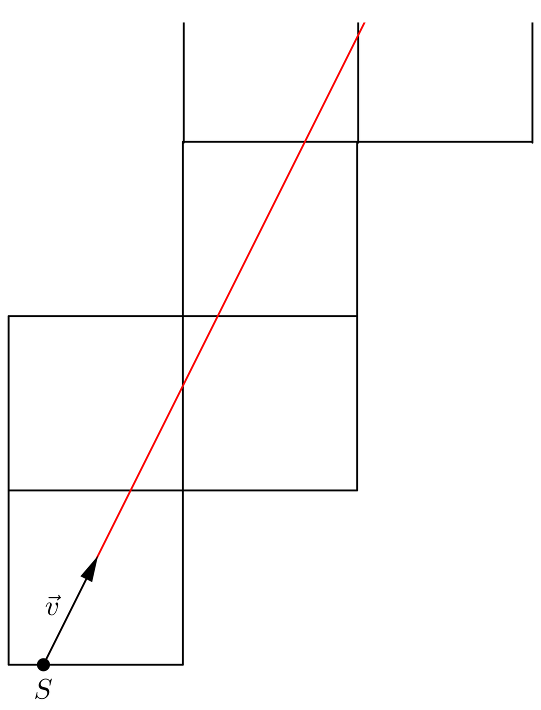


Figure 1

Now we define a set of points on the boundary of the net, which we call K^p. Let P be an intersection between the billiard trajectory in the net and the net itself. P is in the set K^p if: (Figure 2):

(1) The edge of P is parallel to the starting edge.
(2) The distance between the starting point and the left vertex of the starting edge is equal to the distance between P and the left vertex of the edge of P.

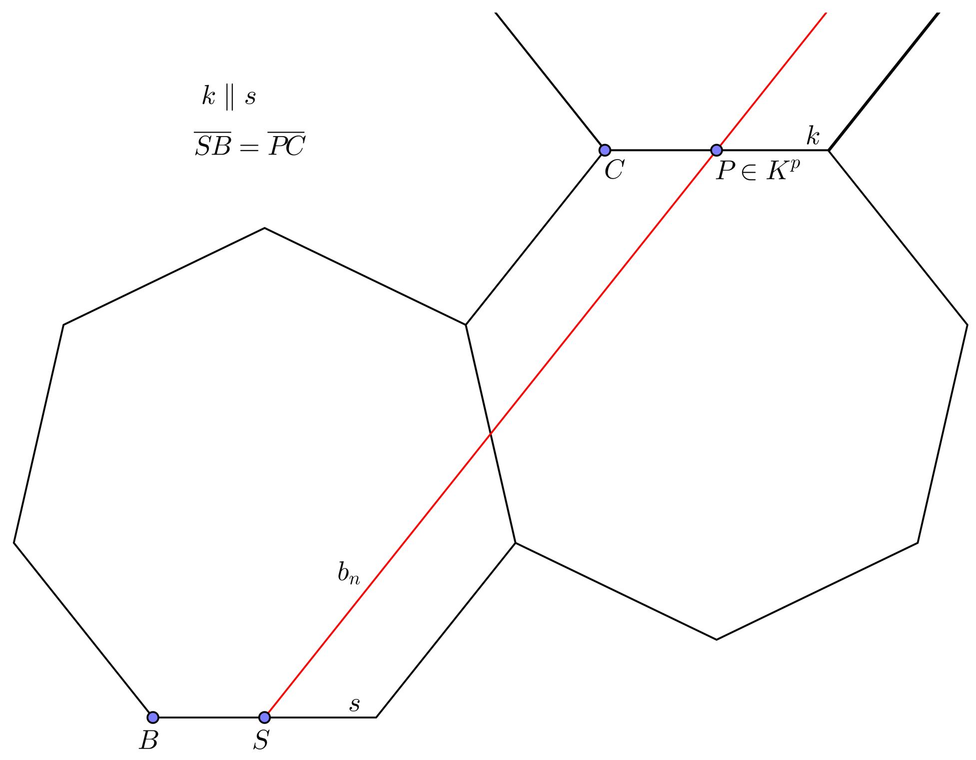


Figure 2

The investigation of the set K^p leads to Theorem 1.

Theorem 1. *A billiard trajectory is periodic if and only if the set K^p of the corresponding billiard trajectory in the net is* ***not*** *empty.*

Every edge of the net is parallel to one edge of the starting polygon. Let's consider a billiard trajectory in the net with a non-empty set K^p in a coordinate system which has its origin at the starting point and the x-axis is parallel to the starting edge. Now we ask the question: Which coordinates can an element of the set K^p have? From the properties of the set K^p and the net, it follows that the x-coordinates are a sum of integer multiples of the cosines of angles of the starting polygon (Figure 3). The y-coordinates are the sum of sines of the same angles. If you divide the y-coordinate over the x-coordinate you get the gradient of the trajectory, which is the tangent of the launching angle. This leads to Theorem 2.

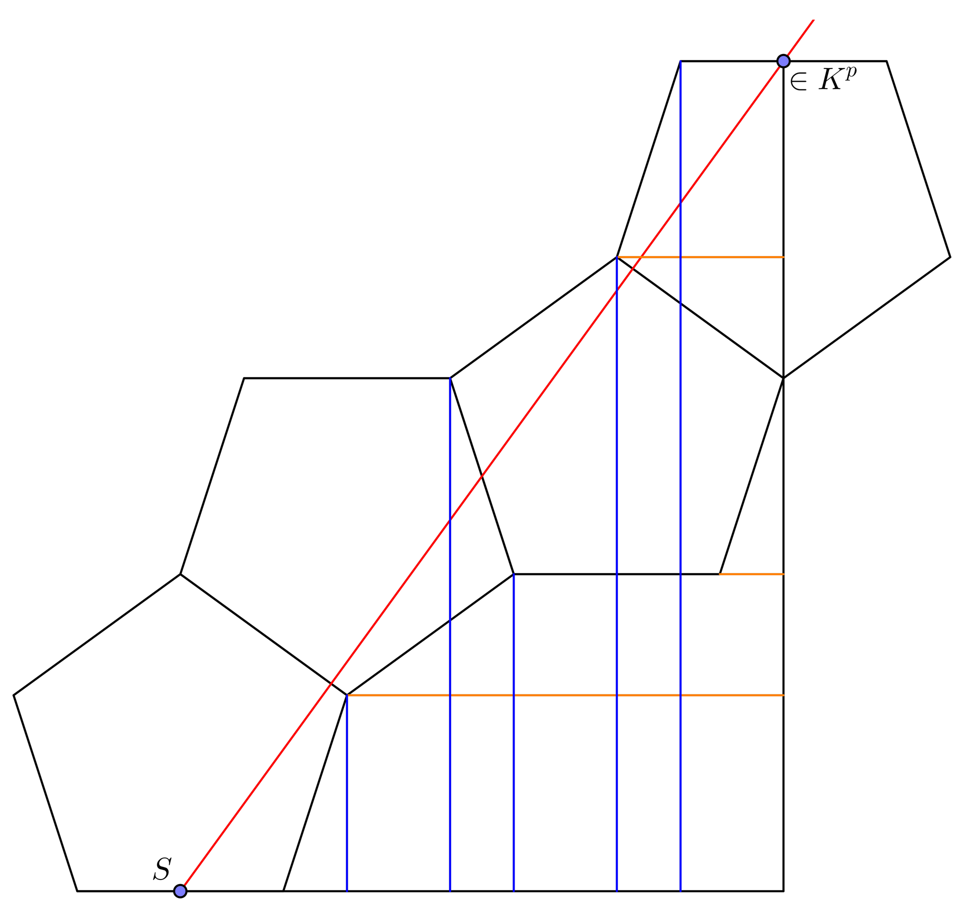


Figure 3