Spectral Gap for Compact Lie Groups

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Notation:

- $G \subset \mathrm{GL}_d(\mathbb{C})$ compact connected simple Lie group (e.g. $SU(d)$)
- μ symmetric probability measure on G

Definition

μ is **non-degenerate** if $\Gamma_\mu = \langle \mathrm{supp}(\mu) \rangle$ is dense in G.

Example: For $g_1, g_2 \in G$,

$$\mu = \frac{1}{4}(\delta_{g_1} + \delta_{g_1^{-1}} + \delta_{g_2} + \delta_{g_2^{-1}}).$$

- Denote

$$\lambda_G(\mu) : L^2(G) \to L^2(G), \qquad f \mapsto \mu * f.$$

- $\lambda_G(\mu)1 = 1$ and $||\lambda_G(\mu)||_{\mathrm{op}} = 1$.
- $L_0^2(G) = \{f \in L^2(G) \ : \ \int f \, d\mathrm{vol}_G = 0\}$

Definition

μ has **spectral gap** if

$$||\lambda_G(\mu)|_{L_0^2(G)}||_{\mathrm{op}} < 1.$$

Conjecture

Spectral gap $\Leftrightarrow$ Non-degenerate

Introduction

If μ has spectral gap, then:

(i) $\exists c > 0$ such that for $f \in C^\infty(G)$,

$$\int f \, d\mu^{*n} = \int f \, dm_G + O(e^{-cn}\mathcal{S}(f)).$$

(ii) $\exists c > 0$ such that for $f_1, f_2 \in L^2(G)$,

$$|\langle \mu^{*n} * f_1, f_2 \rangle - \langle f_1, 1\rangle\langle 1, f_2\rangle| \leq e^{-cn}||f_1||_2||f_1||_2.$$

(iii) From (ii), it follows for $\varepsilon > 0$,

$$\mu^{*n}(B_\varepsilon(e)) \ll \varepsilon^{\dim G} + e^{-cn}.$$

(iv) $\exists c_1, c_2 > 0$ such that

$$\mu^{*n}(B_{e^{-c_1 n}}(e)) \leq e^{-c_2 n}.$$

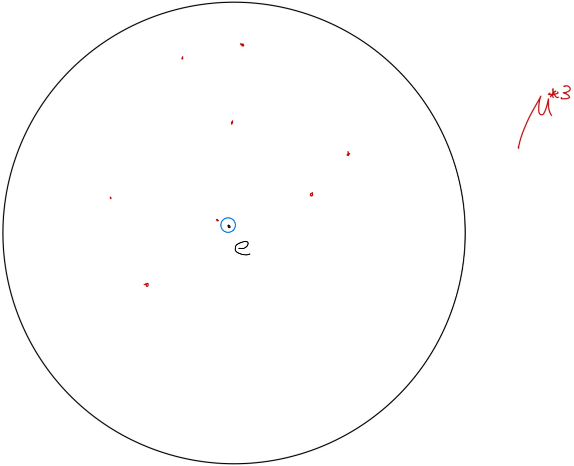
Equally, if μ has spectral gap,

$$\sup_{H<G} \mu^{*n}(B_{e^{-c_1 n}}(H)) \leq e^{-c_2 n}. \tag{1}$$

Definition

A measure satisfying (1) for some $c_1, c_2 > 0$ is called **weakly Diophantine**.

Introduction



Main Results

Theorem 1 (Bourgain-Gamburd 2011 for $SU(d)$, Benoist-de Saxcé 2015)

Spectral gap $\Leftrightarrow$ Weakly Diophantine

Theorem 2 (Bourgain-Gamburd 2011 for $SU(d)$, Benoist-de Saxcé 2015)

Assume μ is:

- Finitely supported and non-degenerate.
- $\mathrm{supp}(\mu) \subset M_d(\overline{\mathbb{Q}})$.

Then μ is weakly Diophantine.

Proof of Theorem 2

For simplicity, $\mu = \frac{1}{4}(\delta_{g_1} + \delta_{g_1^{-1}} + \delta_{g_2} + \delta_{g_2^{-1}})$. Then (Kesten)

$$\mu^{*n}(e) \leq e^{-c_1 n}. \tag{1}$$

Assume g_1, g_1^{-1}, g_2 and g_2^{-1} in $\mathrm{GL}_d(\mathbb{C})$ have rational entries with denominator $\leq q$. Then:

- $g_1 g_2$ has entries of denominator $\leq q^{2d}$.
- If w is a word of length $n \geq 2$ then the denominator of all the entries of $w(g_1, g_2)$ is $\leq (q^{2d})^n$.
- For $c_2 \gg 1$,
$$\mathrm{supp}(\mu^{*n}) \cap B_{e^{-c_2 n}}(e) = \{e\} \tag{2}$$
- (1) + (2):
$$\mu^{*n}(B_{e^{-c_2 n}}(e)) = \mu^{*n}(e) \leq e^{-c_1 n}.$$

Proof of Theorem 2

Proposition

Let μ be non-degenerate. Then $\exists c_1 > 0$ such that for $H < G$,

$$\mu^{*n}(H) \leq e^{-c_1 n}.$$

Proposition

Let μ be finitely supported with $\mathrm{supp}(\mu) \subset M_d(\overline{\mathbb{Q}})$. For $n \gg 1$:
For each $H < G$ there is $H \leq H' < G$ such that

$$\mathrm{supp}(\mu^{*n}) \cap B_{e^{-c_1 n}}(H) \subset H'.$$

Proof of Theorem 1

Aim: Weakly Diophantine $\Rightarrow$ Spectral Gap.

Bourgain-Gamburd method for expansion:

- Expansion condition: μ^{*n} flat for $n \geq C \log |G|$.

To show condition, two steps:

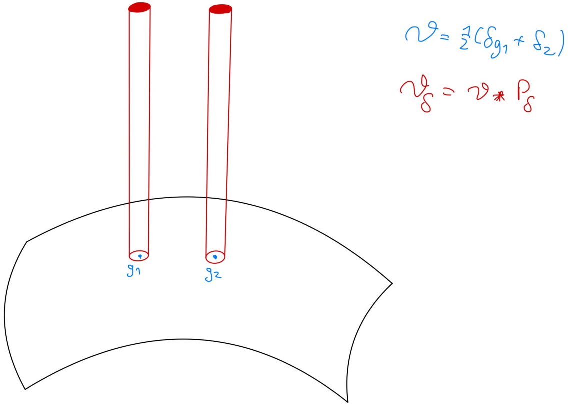
(i) Non-concentration.

(ii) Flattening Lemma.

For compact groups, we need multiscale argument.

Proof of Theorem 1

- Denote
$$P_\delta = \frac{1_{B_\delta}(e)}{\mathrm{vol}(B_{\delta(e)})}.$$
- For measure ν, denote $\nu_\delta = \nu * P_\delta$.



Proof of Theorem 1

Usually for δ small,

$$||\mu_\delta||_2 \asymp \delta^{-\dim(G)}.$$

Lemma (Spectral Gap Condition)

Let μ be non-degenerate. Assume for δ small enough,

$$||(\mu^{*n})_\delta||_2 \leq \delta^{-\frac{1}{4}}$$

for $n \geq C \log(\frac{1}{\delta})$. Then μ has a spectral gap.

Proof of Theorem 1

- $$L^2(G) = \bigoplus_{\pi \in \widehat{G}} \dim(\pi)\pi.$$

- $$||\lambda_G(\mu)|_{L_0^2(G)}|| = \sup_{\pi \in \widehat{G} \setminus \{1_G\}} ||\pi(\mu)||.$$

- For non-degenerate μ and π fixed, $||\pi(\mu)|| < 1$.
- For $G = SU(2)$ and $f \in L^2(G)$,

$$||f||_2^2 = \sum_{k \geq 1} k||\pi_k(f)||_{\mathrm{HS}}^2.$$

Proof of Theorem 1

Lemma (Spectral Gap Condition)

Let μ be non-degenerate. Assume for δ small enough,

$$||(\mu^{*n})_\delta||_2 \leq \delta^{-\frac{1}{4}}.$$

for $n \geq C\log(\frac{1}{\delta})$. Then μ has a spectral gap.

Proof for $SU(2)$:

$$\delta^{-1/2} \geq ||(\mu^{*C\log(\frac{1}{\delta})})_\delta||_2^2 = \sum_{k\geq 1} k||\pi_k(\mu)^{*C\log(\frac{1}{\delta})}\pi_k(P_\delta)||_{HS}^2$$

As $||\pi_k(P_\delta) - \mathrm{Id}_k|| \leq k\delta$, we can omit the term $\pi_k(P_\delta)$ for $k \sim \delta^{-1}$. Thus

$$||\pi_k(\mu)||^{2C\log(\frac{1}{\delta})} \ll k^{-1}\delta^{-1/2} = \delta^{1/2}.$$

- Initially: $||\mu_\delta||_2 \asymp \delta^{-\dim G}$.
- Aim: $||\mu_\delta^{*n}||_2 \asymp \delta^{-1/4}$ for $n \geq C \log(\frac{1}{\delta})$.
- Flattening Lemma: Under suitable assumptions

$$||\mu_\delta * \mu_\delta||_2 \leq \delta^{\varepsilon}||\mu_\delta||_2.$$

Proof of Theorem 1

Expansion for $\mathrm{SL}_2(\mathbb{F}_p)$ relies on:

Product Theorem (Helfgott)

$\exists \varepsilon > 0$ such that for any $A \subset G = \mathrm{SL}_2(\mathbb{F}_p)$ one of the following holds:

(i) (Expansion) $|AAA| \geq |A|^{1+\varepsilon}$.

(ii) (Close to G) $|A| \geq |G|^{1-O(\varepsilon)}$.

(iii) (Trapping) $A \subset H$ for $H \leq G$.

Proof of Theorem 1

Sum-Product Theorem (Erdös-Szemeredi 1983)

There is $\varepsilon > 0$ such that for any **finite** $A \subset \mathbb{R}$,

$$|A + A| + |A \cdot A| \gg |A|^{1+\varepsilon}.$$

- Aim: Generalize to arbitrary subsets of $\mathbb{R}$.
- Idea: Consider A at scale δ.
- Replace $|A|$ by covering number $N(A, \delta)$.

Proof of Theorem 1

- Aim: $N(A+A,\delta)+N(A\cdot A,\delta)\geq \delta^{-\varepsilon}N(A,\delta)$.
- Necessary: $N(A,\delta)\sim\delta^{-\sigma}$, i.e.

$$\delta^{-(\sigma-\varepsilon)}\leq N(A,\delta)\leq \delta^{-(\sigma+\varepsilon)}. \tag{1}$$

- Counterexample: $A=[0,s]$ for any $s>0$.
- Non-concentration on higher scales than δ:

$$N(A\cap B(x,\rho),\delta)\leq \delta^{-\varepsilon}\rho^{\sigma}N(A,\delta), \tag{2}$$

for all $x\in A$ and $\rho\geq\delta$.

Discretized Sum-Product Theorem (Bourgain 2003)

For $\sigma\in(0,1)$ there is $\varepsilon=\varepsilon(\sigma)>0$ such that the following holds for δ small enough.
Let $A\subset[0,1]$ and assume (1) and (2). Then

$$N(A+A,\delta)+N(A\cdot A,\delta)\geq \delta^{-\varepsilon}N(A,\delta).$$

Proof of Theorem 1

Theorem (Bourgain 2003)

For $\sigma \in (0,1)$ there is $\varepsilon = \varepsilon(\sigma) > 0$. Let $A \subset [0,1]$ with $\dim_H(A) \in [\sigma - \varepsilon, \sigma + \varepsilon]$. Then

$$\max(\dim_H(A+A), \dim_H(A \cdot A)) \geq \dim_H(A) + \varepsilon.$$

Corollary (Erdös-Volkmann Ring conjecture, Edgar-Miller Theorem)

Let $S \subset \mathbb{R}$ be a Borel measurable subring. Then either $\dim_H(S) = 0$ or $S = \mathbb{R}$.

Proof of Theorem 1

Discretized Product Theorem (BG 2011 for $SU(d)$, De Saxcé 2015)

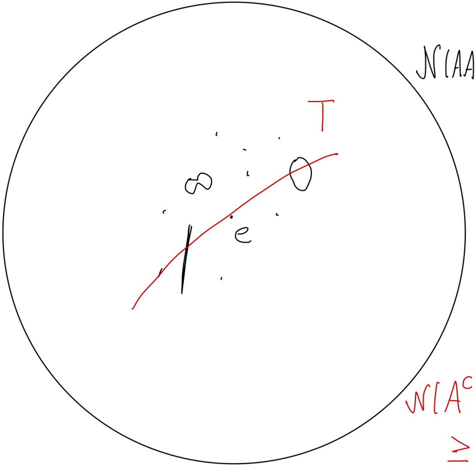
$\exists U$ neighbourhood of the identity such that for any $\sigma \in (0, \dim G)$ there exists $\varepsilon = \varepsilon(\sigma)$ satisfying the following.
Let $A \subset U$ be a subset with:

- $\delta^{-(\sigma-\varepsilon)} \leq N(A,\delta) \leq \delta^{-(\sigma+\varepsilon)}$.
- Not concentrated on higher scales than δ.

Then one of the following holds

(i) (Expansion) $N(AAA,\delta) \geq \delta^{-\varepsilon} N(A,\delta)$

(ii) (Trapping) $A \subset B_{\delta^\varepsilon}(H)$.



$$\mathcal{N}(AAA,\delta) \le \delta^{-\varepsilon}\mathcal{N}(A,\delta)$$

$$\mathcal{N}(A^c \cap T^{(\delta^{1-O(\varepsilon)})},\delta) \ge \delta^{\varepsilon}\mathcal{N}(A,\delta)^{\frac{1}{\dim G}}.$$

Proof of Theorem 1

Sketch of proof: Proof by contradiction.

(i) Assume A is not trapped and has small tripling.

(ii) There is $a \in A^C$ such for C_a the conjugacy class of a,

$$N(A^C \cap C_a, \delta) \leq \delta^{-C\varepsilon} N(A, \delta)^{1-\frac{1}{d}}$$

(iii) There exists a maximal torus T of G such that

$$N(A \cap T^{(\delta^{1-C\varepsilon})}, \delta) \geq \delta^{C\varepsilon} N(A, \delta)^{\frac{1}{d}}.$$

(iv) Growth inside the torus.

(v) Growth in all directions.

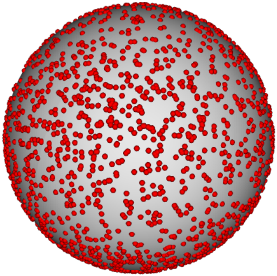
Proof of Theorem 1

This concludes Weakly Diophantine $\Rightarrow$ Spectral Gap:

(i) Product Theorem classifies approximate subgroups at scale δ.

(ii) Scaled Balog-Szemeredi-Gowers: Either convolution flat or supported on translate of approximate subgroup.

(iii) Weak Diophantine property avoids trapping.



Thank you!