The Iwasawa Decomposition for Symmetric Spaces of Non-Compact Type

Let M be a symmetric space of non-compact type and denote $G = \mathrm{Iso}(M)^\circ$. Fix $p \in M$ and denote $K = \mathrm{Stab}_G(p)$. Consider the map

$$\sigma : G \to G, \qquad g \mapsto s_p g s_p^{-1},$$

where s_p is the geodesic symmetry around p. This makes (G, K, σ) into a Riemannian-symmetric pair.

We have the *Cartan involution*

$$\theta := D_e\sigma : \mathfrak{g} \to \mathfrak{g}$$

and the Cartan decomposition

$$\mathfrak{g} = E_1(\theta) \oplus E_{-1}(\theta) = \mathfrak{k} \oplus \mathfrak{p}$$

and consider the positive definite symmetric bilinear form

$$\langle X, Y\rangle = -B_{\mathfrak{g}}(X, \theta(Y)).$$

This bilinear form was the basis for our discussion of roots and root systems. Fix a maximal abelian subspace $\mathfrak{a}$. Recall the following definition and theorem:

Definition. An element $X \in \mathfrak{p}$ is called *regular* if $Z(X) \cap \mathfrak{p}$ is abelian.

Theorem. *For every maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ there exists $X \in \mathfrak{p}$ such that*

$$\mathfrak{a} = Z(X) \cap \mathfrak{p}.$$

In particular, X is regular. Furthermore for any two maximal abelian subspaces $\mathfrak{a}, \mathfrak{a}' \subset \mathfrak{p}$ there is $k \in K$ such that

$$\mathrm{Ad}_G(k)(\mathfrak{a}) = \mathfrak{a}'.$$

This Theorem is proved for symmetric spaces of compact type by using that the Lie subgroup associated to $\mathfrak{a}$ is in fact a torus and so we can find an element in $\mathfrak{p}$ such that the geodesic in the direction of this element is dense. Then one easily sees that this element commutes with $\mathfrak{a}$. One then traverses to symmetric spaces of non-compact type by using the duality between symmetric spaces of compact and non-compact type.

Note that $\{\mathrm{ad}(H) : H \in \mathfrak{a}\}$ is simultaneously diagonalizable so for $\lambda \in \mathfrak{a}^*$ we write

$$0 \neq \mathfrak{g}_\lambda = \{X \in \mathfrak{g} : (\mathrm{ad}(H))X = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}.$$

By choosing a basis we can define Σ^+ the set of positive roots and denote

$$\mathfrak{n} = \oplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda.$$

Note $\mathfrak{n}$ is a nilpotent Lie algebra as $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. and there are only finitely may λ such that $\mathfrak{g}_\lambda \neq 0$.

Theorem. *(Iwasawa decomposition) With the above notations we can write*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Denote by A the Lie group associated to $\mathfrak{a}$ and by N the Lie group associates to $\mathfrak{n}$. Then the groups A and N are simply connected and furthermore the multiplication map

$$K \times A \times N \longrightarrow G, \qquad (k, a, n) \longmapsto kan$$

is a diffeomorphism.

Remark. The Iwasawa decomposition holds for all semisimple Lie groups with the same proof. The proof of the Lie algebra decomposition just uses symmetric positive definite bilinear form

$$\langle X, Y \rangle = -B(X, \theta Y).$$

As one can find such a positive definite bilinear form on any semisimple Lie algebra, the proof then carries over.

Fix $x \in M(\infty)$ and let $X \in \mathfrak{p}$ be the unit vector such that $\exp(tX).p = \gamma_{xp}$, where γ_{xp} is the unique unit speed geodesic with $\gamma_{px}(0) = p$ that is asymptotic to x. Recall that we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{p} & \xrightarrow{D_e\pi} & T_pM \\ \big\downarrow{\scriptstyle \exp} & & \big\downarrow{\scriptstyle \exp_p} \\ G & \xrightarrow{\pi} & M \end{array}$$

This commutative diagram holds as we can associate to each geodesic γ the following one-parameter subgroup of transvections $\mathcal{T}_t = s_{\gamma(t/2)\gamma(0)}$ and then using the fact that one parameter subgroups are precisely given by the exponential map. So there exists a unique $X \in \mathfrak{p}$ such that $\gamma_{px}(t) = \exp(tX).p$. Then we consider the parabolic subgroup at x

$$\begin{aligned} G_x &= \{g \in G \,:\, g \circ \gamma \in [x] \text{ for all } \gamma \in [x]\} \\ &= \{g \in G \,:\, \lim_{t\to\infty} \exp(-tX) g \exp(tX) \text{ exists}\} \end{aligned}$$

We hence have a well defined map

$$T_x : G_x \to G, \qquad g \mapsto \exp(-tX) g \exp(tX).$$

Proposition. *The map T_x is a continuous idempotent homomorphism.*

Proof. Let $g, h \in G_x$, then we have that

$$\begin{aligned} T_x(gh) &= \lim_{t\to\infty} \exp(-tX) gh \exp(t_X) \\ &= \lim_{t\to\infty} \exp(-tX) g \exp(tX) \exp(-tX) h \exp(t_X) = T_x(g) T_x(h), \end{aligned}$$

so T_x is a group homomorphism. To show that T_x is idempotent we note that

$$\begin{aligned} T_x(T_x(g)) &= \lim_{t\to\infty} \exp(-tX) T_x(g) \exp(t_X) \\ &= \lim_{t\to\infty} \lim_{s\to\infty} \exp(-tX) \exp(-sX) g \exp(sX) \exp(tX) \\ &= \lim_{t\to\infty} \lim_{s\to\infty} \exp(-(s+t)X) g \exp((s+t)X) = T_x(g). \end{aligned}$$

We will prove later that T_x is continuous. □

We now introduce the subgroups needed in the more general Iwasawa decomposition:

$$\begin{aligned} N_x &:= \ker(T_x), \\ Z_x &:= \operatorname{im}(T_x), \\ K_x &:= K \cap Z_x, \\ A_x &:= \exp(Z(X) \cap \mathfrak{p}). \end{aligned}$$

We are now ready to state the main theorem of this talk.

Theorem. *(Generalized Iwasawa decomposition) We have diffeomorphisms given by multiplication*

$$K_x \times A_x \times N_x \to G_x$$

and

$$K \times A_x \times N_x \to G.$$

Remark. Note that A_x is not necessarily a subgroup by the above. However, if $X \in \mathfrak{p}$ is regular then $Z(X) \cap \mathfrak{p}$ is maximal abelian and the above is precisely the normal Iwasawa decomposition.

Example. Let $\mathbb{H}$ be the upper half plane and let $p = i \in \mathbb{H}$. Then we have that $G = \mathrm{PSL}_2(\mathbb{R})$ and $K = \mathrm{PSO}_2(\mathbb{R})$. We consider $x \in \mathbb{H}(\infty)$ to we boundary point at infinity which is above p and then we have that X is the unit vector at i that points upwards. So $\gamma_{i\infty}(t) = e^t \cdot i$ and so

$$\gamma_{i\infty}(t) = \exp\left(t\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\right).i.$$

So

$$X = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

and for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$ we have

$$\exp(-tX) g \exp(tX) = \begin{pmatrix} a & e^{-t}b \\ e^t c & d \end{pmatrix}$$

so

$$G_x = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \text{ for } a \in \mathbb{R}\backslash\{0\} \text{ and } b \in \mathbb{R} \right\} / \{\pm I_2\}$$

and

$$\begin{aligned} N_x &= \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \text{ for } b \in \mathbb{R} \right\} / \{\pm I_2\}, \\ Z_x &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \text{ for } a \in \mathbb{R}\backslash\{0\} \right\} / \{\pm I_2\}. \end{aligned}$$

Then

$$K_x = \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \} / \{\pm I_2\} \quad \text{and } A_x = Z_x.$$

Example. Now consider $M = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ and $G = \mathrm{PSL}_n(\mathbb{R})$ and $K = \mathrm{PSO}_n(\mathbb{R})$ then we have that

$$\theta : \mathfrak{g} \to \mathfrak{g}, \qquad X \mapsto -X^T$$

and the Cartan decomposition

$$\mathfrak{g} = E_1(D_e\sigma) \oplus E_{-1}(D_e\sigma) = \mathfrak{k} \oplus \mathfrak{p}$$

with

$$\mathfrak{k} = \{X \in \mathfrak{sl}_n(\mathbb{R}) \,:\, X + X^T = 0\}$$
$$\mathfrak{p} = \{X \in \mathfrak{sl}_n(\mathbb{R}) \,:\, X = X^T\}.$$

So we take some element $X \in \mathfrak{p}$. By some base change we can assume that X is a diagonal matrix. So write $X = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \ldots \geq \lambda_n$. Note that X is regular if and only if all the λ are distinct. However we don't assume that for now. Without loss of generality we say that $\lambda_1 \geq \ldots \geq \lambda_n$. Note then that

$$(\exp(-tX)g\exp(tX))_{ij} = e^{t(\lambda_j - \lambda_i)}g_{ij}.$$

So we have that

$$G_x = \{g \in \mathrm{PSL}_n(\mathbb{R}) \,:\, g_{ij} = 0 \text{ if } \lambda_j > \lambda_i\}$$

and we have

$$Z_x = \{g \in \mathrm{PSL}_n(\mathbb{R}) \,:\, g_{ij} = 0 \text{ if } \lambda_j \neq \lambda_i\}.$$

For $n = (n_{ij}) \in N_x$ we have that

$$e^{t(\lambda_j - \lambda_i)} \to \delta_{ij}.$$

So we get the following characterization of elements of N_x:

1. If $i = j$, then $n_{ii} = 1$.
2. If $i > j$, then $\lambda_i \geq \lambda_j$ and so $n_{ij} = 0$
3. if $i < j$, then $\lambda_i \leq \lambda_j$. If $\lambda_i = \lambda_j$, then $n_{ij} = 0$.

The centralizer of $X \in \mathfrak{p}$ the turns out to be

$$Z(X) \cap \mathfrak{p} = \{A \in \mathfrak{p} \,:\, a_{ij} = 0 \text{ if } \lambda_i \neq \lambda_j\}$$

To be even more explicit let $n = 3$ and consider $X = (1, 1, -2)$. Then we have that

$$G_x = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{SL}_3(\mathbb{R}) \right\},$$
$$Z_x = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{SL}_3(\mathbb{R}) \right\},$$
$$N_x = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SL}_3(\mathbb{R}) \right\},$$
$$Z(X) \cap \mathfrak{p} = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathfrak{sl}_3(\mathbb{R}) \right\}.$$

We now prove the first part of the Iwasawa decomposition, namely that $G_x = K_x A_x N_x$.

Proof. (First Part of Iwasawa decomposition) The proof comprises several steps:

1. $G_x = Z_x N_x$, where the decomposition is unique. To see this note that any $g \in G_x$ can be written as
$$g = T_x(g) T_x(g)^{-1} g,$$
where $T_x(g) \in Z_x$ and $T_x(g)^{-1} g \in N_x$ as T_x is idempotent. To see uniqueness assume $g = zn$ then $T_x(g) = z$ and so z is determined by the choice of g. So also n is unique.

   This in fact also shows that T_x is continuous: It suffices to check continuity at $e \in G_x$. So let $g_i \to e$ as $n \to \infty$. Then we have by the last corollary
$$g_i = z_i n_i$$
with $i \to \infty$. By uniqueness of the decomposition it follows that $z_i \to e$ and $n_i \to e$. So we have
$$T_x(g_i) = z_i \to e = T_x(e).$$
Hence T_x is continuous.

2. $Z_x = \{g \in G_x : \exp(-tX) g \exp(tX) = g \text{ for all } t \in \mathbb{R}\}$. To see this let $g \in Z_x$. Then $g = T_x(h)$ for $h \in Z_x$. Thus we have that $Z_x \subset \mathrm{im}(T_x)$. Conversely if $y \in \mathrm{im}(T_x)$, then we have that some $g \in G_x$ such that $T_x(g) = y$. Then note that for any $s \in \mathbb{R}$
$$\begin{aligned} \exp(-sX) y \exp(sX) &= \exp(-sX) T_x(g) \exp(sX) \\ &= T_x(\exp(-sX)) T_x(g) T_x(\exp(sX)) \\ &= T_x(\exp(-sX) g \exp(sX)) \\ &= \lim_{t \to \infty} \exp(-(s+t)X) \end{aligned}$$
so indeed $y \in Z_x$.

3. $A_x \subset Z_x$: To see this note that for $Y \in Z(X) \cap \mathfrak{p}$ we have that $[tX, Y] = 0$ for all $t \in \mathbb{R}$ and hence
$$\exp(Y) \exp(tX) = \exp(Y + tX) = \exp(tX + Y) = \exp(tX) \exp(Y).$$
So indeed $\exp(Y) \in Z_x$.

4. Let $g \in Z_x$. Then there exists $a \in A_x$ with $g(p) = a(p)$. To see this let Y be such that $\exp(Y).p = g.p$. Then X and Y span a flat strip and so $[X, Y] = 0$ as the curvature is zero.

5. We show $Z_x = K_x A_x$: By taking inverses we can also show that $Z_x = A_x K_x$ For $z \in Z_x$ choose by the first step $a \in A_x$ with $z(p) = a(p)$. Then we have that $k = a^{-1} z \in Z_x \cap K$ and so we have $z = ak$. Next we prove uniqueness. To see this choose write $z = ak = a'k'$ for $a, a' \in A_x$ and $k, k' \in K_x$. Then we have that there are $Y, Y' \in \mathfrak{p} \cap Z(X)$ such that $\exp(Y)(p) = z(p) = \exp(Y')(p)$. As a symmetric space of non-compact type is simply connected, we have that $\exp : T_p M \to M$ is a diffeomorphism. So we conclude that $Y = Y'$ and so we are done.

□

Proof. (of (ii) of the Iwasawa decomposition) Recall that the action of G_x on M is transitive. So given $g \in G$ choose $g_x \in G_x$ such that $g(p) = g_x(p)$ so $g_x^{-1} g \in K$ and $G = KG_x$. So by part (i) $G = KK_xA_xN_x = KA_xN_x$ so we have existence.

It remains to prove uniqueness. Assume for the moment that $K \cap A_x^2 N_x = \{\mathrm{id}\}$. Write $g \in KA_xN_x = G$ and write $g = k_1a_1n_1 = k_2a_2n_2$ with $k_1, k_2 \in K$, $a_1, a_2 \in A_x$ and $n_1, n_2 \in N_x$. So

$$k_1^{-1}k_2 = a_1n_1n_2^{-1}a_2^{-1} = a_1a_2^{-1}(a_2n_1n_2^{-1}a_2^{-1}) \in K \cap A_x^2 N_x = \{\mathrm{id}\}$$

as N is normal. So $k_1 = k_2$ and $a_1n_1 = a_2n_2$. Applying T_x on both sides we have $a_1 = a_2$ so we are done.

So it remains to show that $K \cap A_x^2 N_x = \{\mathrm{id}\}$. The proof again comprises several steps:

1. $K \cap N_x = \{\mathrm{id}\}$: Let $g \in K \cap N_x$ so $\exp(-tX)g\exp(tX) \to \mathrm{id}$ as $t \to \infty$. Thus
$$\mathrm{Ad}(\exp(-tX))\mathrm{Ad}(g)\mathrm{Ad}(\exp(tX)) \to \mathrm{id}$$
as $t \to \infty$. Thus the characteristic polynomial of $\mathrm{Ad}(g)$ is $(T-1)^{\dim(\mathfrak{g})}$. So $\mathrm{Ad}(g)$ is unipotent by Caley-Hamilton. Furthermore we claim that $\mathrm{Ad}(g)$ preserves the inner product $\langle X, Y\rangle = -B_{\mathfrak{g}}(X, \theta Y)$. To see this note that $\sigma \circ \mathrm{int}(g) = \mathrm{int}(g) \circ \sigma$ as $\sigma(g) = g$ and so by passing to the derivative we have $\mathrm{Ad}(g)\theta = \theta\mathrm{Ad}(g)$. Furthermore the Killing form is $\mathrm{Ad}(g)$ invariant, so we conclude:
$$\begin{aligned}\langle \mathrm{Ad}(g)X, \mathrm{Ad}(g)Y\rangle &= -B_{\mathfrak{g}}(\mathrm{Ad}(g)X, \theta\mathrm{Ad}(g)Y)\\ &= -B_{\mathfrak{g}}(\mathrm{Ad}(g)X, \mathrm{Ad}(g)\theta Y) = \langle X, Y\rangle.\end{aligned}$$
As every orthogonal matrix is diagonalizable, and a unipotent matrix has only 1 as eigenvalues, we conclude that $\mathrm{Ad}(g) = \mathrm{id}$. So $\mathrm{Ad}(g) = \mathrm{id}$, implying that $g = \mathrm{id}$ as Ad is injective.

2. $K \cap A_x^2 = \{\mathrm{id}\}$: Let $g = \exp(Y)\exp(Y') \in K \cap A_x^2$ for $Y, Y' \in Z(X) \cap \mathfrak{p}$. Then
$$p = g.p = \exp(Y)\exp(Y').p$$
and so $\exp(-Y).p = \exp(Y').p$. So $\mathrm{Exp}_0(D_e\pi - Y) = \mathrm{Exp}_0(D_e\pi Y')$ and so as the exponential map is a diffeomorphism we conclude $D_e\pi - Y = D_e\pi Y'$ and so $-Y = Y'$ and hence $g = \mathrm{id}$.

3. $K_x = K \cap Z_x = K \cap G_x$: Note that we already have $K_x = K \cap Z_x \subset K \cap G_x$ since $Z_x \subset G_x$. Now let $g \in K \cap G_x$. Note that $g\exp(tX).p$ is a unit speed geodesic as g asymptitic to x as g is an isometry with $gx = x$. Hence by uniqueness we have that
$$g\exp(tX).p = \gamma_{xp}(t) = \exp(tX).p.$$
So
$$\exp(-tX)g\exp(tX) \in K$$

for all t. As $g \in G_x$ we have that $T_x(g) = \lim_{t\to\infty} \exp(-tX)g\exp(tX)$ exists. Thus $T_x(g) \in K$ as K is closed. As T_x is idempotent we thus have that

$$T_x(T_x(g^{-1})g) = \mathrm{id}$$

and hence $T_x(g^{-1})g \in K \cap N_x$. Wir haben nun $g = T_x(g) \in \mathrm{im}(T_x) = Z_x$.

4. $K \cap A_x^2 N_x = \{\mathrm{id}\}$: Let $\phi \in K \cap A_x^2 N_x \subset K \cap G_x = K \cap Z_x$ so $\phi = a_1 a_2 n$ with $a_1, a_2 \in A_x$ and $n \in N_x$. Since $g \in Z_x$ we have

$$T_x(g) = a_1 a_2 = g = a_1 a_2 n$$

so $n = \mathrm{id}$ and $g \in K \cap A_x^2 = \{\mathrm{id}\}$. So we are done.

□

Lemma. *The group N_x is a path connected closed normal subgroup of G_x.*

Proof. As T_x is a continuous homomorphism, we have that $N_x = T^{-1}(0)$ is a closed normal subgroup. To see that N_x is path connected, note that for any $g \in N_x$ and $t \in \mathbb{R}$ we have that

$$\exp(-tX)g\exp(tX) \in N_x.$$

Thus we can consider the path

$$\gamma : [0,\infty) \to N_x, \qquad t \mapsto \exp(-tX)g\exp(tX).$$

As $\lim_{t\to\infty} \gamma(t) = e_G$ we can reparametrize and extend γ to a path from g to $e \in N_x$. Thus G_x is path connected. □

Denpendence on x and p? If you change p, you only conjugate everything. If you change x, it is not so easy to understand what happens.