Random Walks on Non-compact Semisimple Lie Groups

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- G non-compact semisimple Lie group such as $\mathrm{SL}_2(\mathbb{R})$.
- Open questions for random walks on G:
    - Furstenberg measure absolutely continuous?
    - Local limit theorem.
    - Effective equidistribution on homogeneous spaces.
- Known if distribution is $\ll \mathrm{vol}_G$
- Goal of talk: Understand for certain finitely supported measures.

Plan of talk:

1. Introduction.
2. Local limit theorem on symmetric spaces.
3. Interlude: Random walks on compact groups (Bourgain-Gamburd method).
4. Effective equidistribution on homogeneous spaces (joint work with Wooyeon Kim)

Introduction: Random walks on $\mathbb{R}$

Let $Z_1, Z_2, \ldots$ be i.i.d. random variables on $\mathbb{R}$. Write

$$Y_n = Z_1 + \ldots + Z_n.$$

Assume $E[Z_i] = 0$ and $\sigma^2 = E[Z_i^2] < \infty$.

Central Limit Theorem: For all $a, b \in \mathbb{R}$ with $a \leq b$,

$$\lim_{n\to\infty} P\left[a \leq \frac{Y_n}{\sqrt{n}} \leq b\right] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b e^{-\frac{x^2}{2\sigma^2}}\, dx.$$

Local Limit Theorem: If the distribution of Z_i is non-lattice,

$$\lim_{n\to\infty} \sqrt{n} \cdot P[a \leq Y_n \leq b] = \frac{(b-a)}{\sqrt{2\pi\sigma^2}}.$$

Equivalent: If $Z_i \sim \mu$, then $Z_1 + \ldots + Z_n \sim \mu^{*n}$ and

$$\sqrt{n} \cdot \mu^{*n} \to \frac{\mathrm{vol}_{\mathbb{R}}}{\sqrt{2\pi\sigma^2}}.$$

Introduction

Local limit theorem for groups actions $G \curvearrowright X$?

Let μ be a probability measure on G. Then

$$\mu^{*n}.\delta_{x_0}$$

is the distribution of the Y_{n,x_0} random walk after n steps starting at $x_0 \in X$.

Aim: Describe for $A \subset X$,

$$P[Y_{n,x_0} \in A] = (\mu^{*n}.\delta_{x_0})(A) = \int 1_A(g.x_0)\, d\mu^{*n}(g).$$

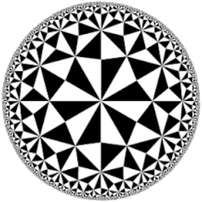
Introduction

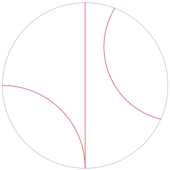
Example: Isometric action of $\mathrm{SL}_2(\mathbb{R})$ on hyperbolic disc

$$\mathbb{D} = \{z \in \mathbb{C} \,:\, |z| < 1\} \quad \text{with metric} \quad 4\frac{dx^2 + dy^2}{(1-(x^2+y^2))^2}.$$

For $g \in \left(\begin{smallmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{smallmatrix}\right) \in \mathrm{SU}(1,1) \cong \mathrm{SL}_2(\mathbb{R})$ and $x \in \mathbb{D}$,

$$g.x = \frac{\alpha x + \overline{\beta}}{\beta x + \overline{\alpha}}.$$





Introduction

$\mathrm{SL}_2(\mathbb{R}) \curvearrowright \mathbb{D}$ and consider $\mu = \frac{1}{2}(\delta_{g_1} + \delta_{g_2})$ for $g_1, g_2 \in \mathrm{SL}_2(\mathbb{R})$.

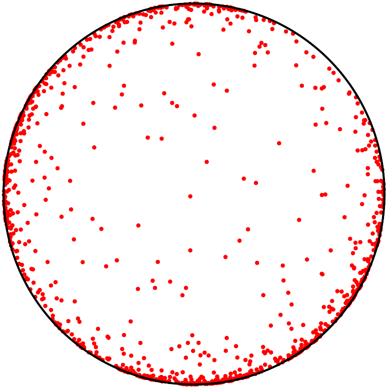


Figure: Support of $\mu^{*10}.0$.

Introduction

- G group (locally compact Hausdorff).
- X locally compact Hausdorff space with Borel measure vol_X.
- $G \curvearrowright X$ continuously, transitively and measure-preservingly.
- μ compactly supported, symmetric (i.e. $\mu = \mu^{-1}$) and aperiodic (i.e. $\mathrm{supp}(\mu) \not\subset gH$ for $H < G$ closed and $g \in G$) probability measure on G.

Local Limit Conjecture

There is a real sequence $(a_n)_{n\geq 1}$ such that for every $x_0 \in X$ there is a continuous function ψ_{μ,x_0} on X satisfying

$$a_n \cdot (\mu^{*n}.\delta_{x_0}) \to \psi_{\mu,x_0} \cdot \mathrm{vol}_X \qquad \text{as } n \to \infty. \tag{1}$$

- Important cases: $X = G$ or X has finite volume.
- When is ψ_{μ,x_0} constant?
- Error rates for (1)? Need number theoretic input.
- Analogy: Ergodic Theorem. (1) implies ratio ergodic theorem.

Introduction

The local limit theorems are known in the following cases:

- Compact groups (Ito-Kawada 1940).
- $\mathbb{R}^d$ and $\mathbb{Z}^d$.
- Simply connected nilpotent groups (Breuillard 2005, Diaconis-Hough 2021, Breuillard-Bénard 2023).
- $\mathrm{Isom}(\mathbb{R}^d) \curvearrowright \mathbb{R}^d$ (Varjú 2015, Lindenstrauss-Varjú 2016).
- Discrete ameanable groups (Avez 1973).
- Free group (Lalley 1993), Discrete hyperbolic group (Gouëzel 2014).

In all the ameanable cases, the limit measure is the Haar measure.

Local Limit Theorem on Symmetric Spaces

Local Limit Theorem on Symmetric Spaces

Theorem (Bougerol 1982)

- G non-compact semisimple Lie group with finite center.
- μ non-degenerate (i.e semi-group generated by support is dense in G)
- μ spread out (i.e $\mu^{*n} \not\perp \mathrm{vol}_G$ for some $n \geq 1$)

Then there is a continuous function ψ_μ on G depending on μ such that

$$\frac{n^{\ell/2}}{\sigma^n}\mu^{*n} \to \psi_\mu \cdot \mathrm{vol}_G \tag{1}$$

as $n \to \infty$, where $\ell \in \mathbb{Z}_{\geq 1}$ depends on G and $\sigma = ||\lambda_G(\mu)|| < 1$.

A finitely supported probability measure is never spread out.

Theorem (K. 2022)

(1) holds on associated symmetric space $X = G/K$ for some finitely supported non-degenerate measures on G.

Local Limit Theorem on Symmetric Spaces

Method: Harmonic Analysis on symmetric space.

Consider an Iwasawa decomposition $G = KAN$ with associated minimal parabolic $P = MAN$. Denote

$X = G/K$	the associated symmetric space, and
$\Omega = G/P = K/M$	the (Furstenberg) boundary.

Denote by $\rho_0 : G \to U(L^2(\Omega))$ the associated unitary Koopman representation of the G action on the boundary Ω.

Consider

$$S_0 = \rho_0(\mu) = \int \rho_0(g)\, d\mu(g).$$

S_0 may be understood as the Fourier transform of μ at 0.

We reduce the local limit theorem on X to understanding spectral properties of S_0.

Local Limit Theorem on Symmetric Spaces

Definition

A bounded operator $S : \mathcal{H} \to \mathcal{H}$ on a Hilbert space is called **quasicompact**, if

$$\rho_{\mathrm{ess}}(S) = \sup_{\lambda \in \sigma(A) \setminus \sigma_{\mathrm{disc}}(A)} |\lambda| < \rho(S) = \sup_{\lambda \in \sigma(A)} |\lambda|.$$

Theorem (K. 2022)

Let μ be a non-degenerate (i.e. $\langle \mathrm{supp}(\mu) \rangle$ is dense in G) probability measure on G with finite second moment. Assume that S_0 is quasicompact. Then there is a continuous function ψ_μ on G depending on μ such that for all $f \in C_c^\infty(X)$,

$$\lim_{n \to \infty} \frac{n^{\ell/2}}{\sigma^n} \int f(g.x_0) \mu^{*n}(g) = \int f(g.x_0) \psi_\mu(g) \, d\mathrm{vol}_G(g)$$

for all $x_0 \in X$ where $\ell \in \mathbb{Z}_{\geq 1}$ depends on G and $\sigma = ||S_0||$.

Local Limit Theorem on Symmetric Spaces

Write $\mathfrak{a} = \mathrm{Lie}(A)$. For $r \in \mathfrak{a}^*$ denote by $\rho_r : G \to U(L^2(\Omega))$ the r-principal series representation of G, i.e. for $f \in L^2(\Omega)$, $g \in G$ and $\omega \in \Omega$,

$$(\rho_r(g)f)(\omega) = \left(\frac{dg.\mathrm{vol}_\Omega}{d\mathrm{vol}_\Omega(\omega)}\right)^{\frac{1}{2} - i \cdot r} f(g^{-1}.\omega).$$

Then for $f \in C_c^\infty(X)$, $r \in \mathfrak{a}^*$ and $\omega \in \Omega$ define

$$\widehat{f}(r, \omega) = (\rho_{-r}(f)1)(\omega) = \int_G f(g)(\rho_{-r}(g)1)(\omega)\, d\mathrm{vol}_G(g).$$

Then for $x \in X$, we have a Fourier inversion formula

$$f(x) = \int_{\mathfrak{a}^*} \int_\Omega \widehat{f}(r, \omega)(\rho_r(g)1)(\omega)\, d\mathrm{vol}_\Omega(\omega) d\nu_{\mathrm{sph}}(r).$$

Local Limit Theorem on Symmetric Spaces

For a probability measure μ on G and $r \in \mathfrak{a}^*$,

$$S_r = \rho_r(\mu) = \int \rho_r(g)\, d\mu(g).$$

Then for $x_0 = h_0.e \in X$ and $f \in C_c^\infty(X)$,

$$\int f(g.x_0)\, d\mu^{*n}(g) = \int_{\mathfrak{a}^*} \int_\Omega \widehat{f}(r,\omega)(S_r^n \rho_r(h_0)1)(\omega)\, d\mathrm{vol}_\Omega(\omega) d\nu_{\mathrm{sph}}(r).$$

We need to understand spectral properties of S_r.

It follows from S_0 being quasicompact:

(1) Perron-Frobenius: $\lambda \in \sigma(S_0)$ with $|\lambda| = ||S_0||$ implies $\lambda = ||S_0||$.

(2) S_r quasicompact for small r.

(3) $\rho(S_r) < ||S_0||$ for $r \neq 0$.

It holds

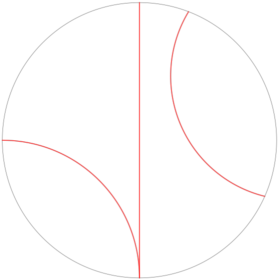
$$\psi_\mu(g) = \langle \eta_0, \rho_0(g)\eta_0' \rangle$$

for suitably normalized functions $\eta_0, \eta_0' \in L^2(\Omega)$ with $S_0\eta_0 = ||S_0||\eta_0$ and $S_0^*\eta_0' = ||S_0||\eta_0'$.

Local Limit Theorem on Symmetric Spaces

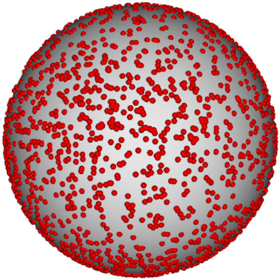
How to understand spectral properties of S_0?

If $\mu = \frac{1}{2}(\delta_{g_1} + \delta_{g_2})$ then $S_0 = \frac{1}{2}(\rho_0(g_1) + \rho_0(g_2))$.



We can understand S_0 in some cases based on the Bourgain-Gamburd (2008) method, generalized to non-compact groups initially by Bourgain (2012) and further generalized by Boutonnet-Ioana-Salehi Golsefidy (2017).

Interlude: Random Walks on Compact Groups



Random Walks on Compact Groups

Local Limit Theorem (Ito-Kawada 1940)

Let G be a compact group and μ be an aperiodic probability measure on G. Then as $n \to \infty$,

$$\mu^{*n} \to \mathrm{vol}_G.$$

Denote $\lambda_G(\mu) : L^2(G) \to L^2(G)$ given for $f \in L^2(G)$ and $x \in G$ by

$$(\lambda_G(\mu)f)(x) = \int f(g^{-1}x)\, d\mu(g).$$

Theorem (Bourgain-Gamburd 2008, Benoist-de Saxcé 2015)

- G compact connected simple Lie group
- μ aperiodic symmetric probability measure
- Assume $\mathrm{Ad}(\mathrm{supp}(\mu)) \subset \mathrm{GL}_d(\overline{\mathbb{Q}})$

Then $\lambda_G(\mu)$ is quasicompact. Moreover, there is $\theta > 0$ depending on μ such that for $f \in \mathrm{Lip}(G)$,

$$\int f(g)\, d\mu^{*n} = \int f\, d\mathrm{vol}_G + O_\mu(\max(\mathrm{Lip}(f), ||f||_\infty)e^{-\theta n}).$$

G compact connected simple Lie group.

Definition

A measure μ on G satisfying for n large enough,

$$\sup_{H<G} \mu^{*n}(B_{e^{-c_1 n}}(H)) \leq e^{-c_2 n}$$

for some $c_1, c_2 > 0$ is called **weakly Diophantine** or (c_1, c_2)-**Diophantine**.

Theorem (Benoist-de Saxcé 2015)

- μ aperiodic symmetric probability measure
- Assume $\mathrm{Ad}(\mathrm{supp}(\mu)) \subset \mathrm{GL}_d(\overline{\mathbb{Q}})$

Then μ is weakly Diophantine.

Random Walks on Compact Groups

Bourgain-Gamburd Method: Understand random walk at scale $\delta > 0$: For a measure ν, denote

$$\nu_\delta(x) = \frac{\nu(B_\delta(x))}{\mathrm{vol}_G(B_\delta(e))}.$$

Measures the *dimension* of μ at scale δ.

Aim: Show $(\mu^{*n})_\delta(x) \approx 1$ for $n \geq C \log \frac{1}{\delta}$.

High dimension is enough: One shows, using that μ is weakly Diophantine,

$$||(\mu^{*n})_\delta||_\infty \leq \delta^{-\frac{1}{4}}$$

for $n \geq C \log \frac{1}{\delta}$.

Main engine: Sum product theorem by Erdös and Szemeredi: There is $\varepsilon > 0$ such that for any finite $A \subset \mathbb{R}$,

$$|A + A| + |A \cdot A| \gg |A|^{1+\varepsilon}.$$

Back to Local Limit Theorem on Symmetric Spaces

Local Limit Theorem on Symmetric Spaces

Notation:

G connected **simple** non-compact Lie group with finite center

$X = G/K$ associated symmetric space

μ probability measure on G

$$S_0 = \rho_0(\mu) = \int \rho_0(g)\, d\mu(g).$$

To apply Bourgain-Gamburd method we need

- $\mathrm{supp}(\mu) \subset B_\varepsilon(e)$,
- and strong Diophantine properties.

Definition

Let $c_1, c_2, \varepsilon > 0$. μ is called (c_1, c_2, ε)-**Diophantine** if

- $\mathrm{supp}(\mu) \subset B_\varepsilon(e)$.
- μ is $(c_1 \log \frac{1}{\varepsilon}, c_2 \log \frac{1}{\varepsilon})$-Diophantine, i.e. for n large enough,

$$\sup_{H<G} \mu^{*n}(B_{\varepsilon^{c_1 n}}(H)) \leq \varepsilon^{c_2 n}.$$

Bourgain (2012) and Boutonnet-Ioana-Salehi Golsefidy 2017: Strong flattening results for these measures:

$$||(\mu^{*n})_\delta||_\infty \leq \delta^{-\gamma}$$

for $n \geq C_\gamma \frac{\log \frac{1}{\delta}}{\log \frac{1}{\varepsilon}}$.

There are many examples of finitely supported (c_1, c_2, ε)-Diophantine probability measures.

Boutonnet-Ioana-Salehi Golsefidy (2017)

- $\Gamma < G$ be a countable dense subgroup.
- $\mathrm{Ad}(\Gamma) \subset \mathrm{GL}_d(\overline{\mathbb{Q}})$.

Then there exist $c_1, c_2 > 0$ such that for every $\varepsilon_0 > 0$ there is $0 < \varepsilon < \varepsilon_0$ and a finitely supported symmetric measure μ satisfying

- $\mathrm{supp}(\mu) \subset \Gamma \cap B_\varepsilon(e)$, and
- μ is (c_1, c_2, ε)-Diophantine.

Theorem (K. 2022)

Let $c_1, c_2 > 0$. Then every symmetric and (c_1, c_2, ε)-Diophantine probability measure μ for small enough ε (depending on c_1, c_2 and G) satisfies that

$$S_0 = \rho_0(\mu) \qquad \text{is quasicompact.}$$

In particular, the local limit theorem holds on X.

Proof uses flattening and further techniques from Bourgain (2012) or Boutonnet-Ioana-Salehi Golsefidy (2017).

Local Limit Theorem on Symmetric Spaces

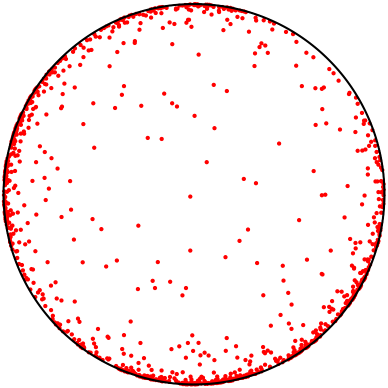


Figure: Distribution of $Y_{10}.0$.

There is a unique μ-stationary probability measure ν_F on $\Omega = G/P$ (i.e. $\mu * \nu_F = \nu_F$) called the Furstenberg measure of μ.

Question: Kaimanovich-Le Prince (2011)

Is the Furstenberg measure of a finitely supported measure always singular to vol_X?

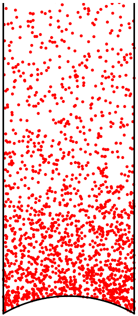
Answer: No:

- Bárány–Pollicott-Simon (2012): Probabilistic Construction.
- Bourgain (2012): Explicit Construction.

Theorem (Léquen 2022, independently K. 2022)

Let $c_1, c_2 > 0$ and $m \in \mathbb{Z}_{\geq 1}$. Then every symmetric and (c_1, c_2, ε)-Diophantine probability measure μ for small enough ε (depending on m,c_1, c_2 and G) has absolutely continuous Furstenberg measure with density in $C^m(\Omega)$.

Effective Equidistribution of Random Walks on Homogeneous Spaces
(joint work with Wooyeon Kim)



- G connected simple Lie group (with finite center).
- $X = G/\Gamma$ for Γ a lattice.
- $\mathrm{supp}(\mu)$ generates dense semi-group of G.

Theorem (Special case of Benoist-Quint)

The Haar probability measure vol_X is the unique μ-stationary measure on X.

Corollary, Local limit theorem (Special case of Bénard 2022)

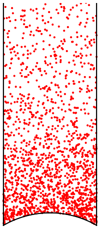
If μ symmetric and $\mathrm{supp}(\mu)$ is countable, then for all $x_0 \in X$, as $n \to \infty$,

$$\mu^{*n}.\delta_{x_0} \to \mathrm{vol}_X. \tag{1}$$

Open problem: Prove error rates for (1).

Consider $\mathrm{ht} : X \to \mathbb{R}_{\geq 1}$ a geometric height function of X:

- Measures how high $x \in X$ is in any cusp of X.
- $\mathrm{inj}(x)^{-O(1)} \ll \mathrm{ht}(x)$.
- If X is compact, then $\mathrm{ht} \equiv 1$.



For $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ the height of $x \in X$ is the imaginary part of the corresponding point in the fundamental domain of $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$.

Using the flattening results and a novel argument passing from high dimension to effective equidistribution on X, we deduce:

Theorem (Wooyeon Kim + K. 2023)

Let $c_1, c_2 > 0$. Then every symmetric (c_1, c_2, ε)-Diophantine probability measure μ on G for small enough ε satisfies the following:
There is $\theta = \theta(\Gamma, \mu) > 0$ such that for every bounded Lipschitz function $f \in \mathrm{Lip}(X)$,

$$\int f(g.x_0)\, d\mu^{*n}(g) = \int f\, d\mathrm{vol}_X + O_{\Gamma,\mu}\left((\mathrm{Lip}(f) + \mathrm{ht}(x_0)||f||_\infty)e^{-\theta n}\right)$$

for all $x_0 \in X$ and $n \geq 1$.

We can prove effective density of $\mu^{*n}.\delta_{x_0}$ for a more general class of measures.

If $\mathrm{supp}(\mu) = S$, then

$$\mathrm{supp}(\mu^{*n}.\delta_{x_0}) = S^n.x_0.$$

Define for $x_0 \in X$,

$$\mathrm{diam}_\varepsilon(X, S, x_0) = \min\{n \geq 0 \,:\, S^n.x_0 \text{ is } \varepsilon\text{-dense in } \{\mathrm{ht} \leq \varepsilon^{-1}\}\}.$$

Theorem (Wooyeon Kim + K. 2023)

- $S \subset G$ symmetric set (i.e. $S = S^{-1}$) generating a dense subgroup of G
- $\mathrm{Ad}(S) \subset \mathrm{GL}_d(\overline{\mathbb{Q}})$

Then for $x_0 \in X$ and $\varepsilon > 0$,

$$\mathrm{diam}_\varepsilon(X, S, x_0) \ll_{\Gamma,S} \log \varepsilon^{-1} + \log \mathrm{ht}(x_0),$$

where the implied constant depends on Γ and S.

Proof uses that by Boutonnet-Ioana-Salehi Golsefidy 2017 there is (c_1, c_2, ε)-Diophantine probability measure μ for arbitrarily small ε with

$$\mathrm{supp}(\mu) \subset S^\ell.$$

Then we apply effective equidistribution for μ.

Summary

For measures close to the identity and satisfying strong Diophantine properties, using flattening results for non-compact groups as initiated by Bourgain we understand:

- Furstenberg measure.
- Local limit theorem on symmetric space.
- Effective equidistribution on homogeneous space.

Thank you!