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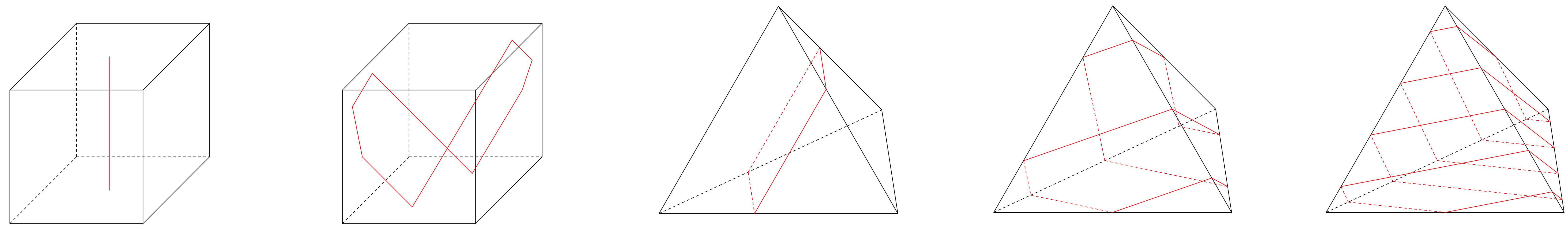
On periodic billiard trajectories in regular polygons and simple closed geodesics on the tetrahedron, cube and octahedron



Theorem 2. *For the launching angle α of a* ***periodic*** *billiard trajectory in a regular polygon with edge number n and $a_i \in \mathbb{Z}$; $b_i \in \mathbb{Z}$ it applies:*

$$\tan(\alpha) = \frac{\sum_{i=0}^{n-1} a_i \cdot \sin(i \cdot \frac{2 \cdot \pi}{n})}{\sum_{i=0}^{n-1} b_i \cdot \cos(i \cdot \frac{2 \cdot \pi}{n})} \tag{1}$$

The symmetric properties of the sine and cosine imply a simplification of Theorem 2 for certain regular polygons. However, the angle condition from Theorem 2 is necessary, but it is **not** proven that it is sufficient, which means that there might exist angles which meet the condition of Theorem 2, but are **not** angles of periodic billiard trajectories. Only for the regular polygons with 3,4 or 6 edges do we know that the condition of Theorem 2 is definitely sufficient. A property of those regular polygons is that they tessellate the plane. Therefore we are able to locate all elements of the set K^p.

Periodic billiard trajectories in the cube

We can transfer the approach of periodic billiard trajectories in regular polygons. We can define the set K^p for the cube and subsequently prove Theorem 1 for the cube. As the square tessellates the plane, the cube fills the 3-dimensional space. This implies that a billiard trajectory in the cube is periodic, if and only if the launching vector is a multiple of a vector which consits of integers. You can see above some periodic billiard trajectories inside the cube.

Geodesics on platonic solids

If we unfold the surface of a platonic solid to the 2-dimensional plane, we get a net consisting of regular polygons. With this approach we can investigate a curve on the surface of a platonic solid by considering the net-depiction of the curve. A curve which forms a straight line in the net is called a geodesic. Since every straight line in the net is the net-depiction of a billiard trajectory in a regular polygon, we have a link between geodesics and billiard trajectories.

A geodesic is called closed if it reapeats itself - like a periodic billiard trajectory. In fact, a geodesic is closed if and only if the set K^p of the net-decpiction of the geodesic is not empty. Figure 4 shows a closed geodesic and the corresponding net.

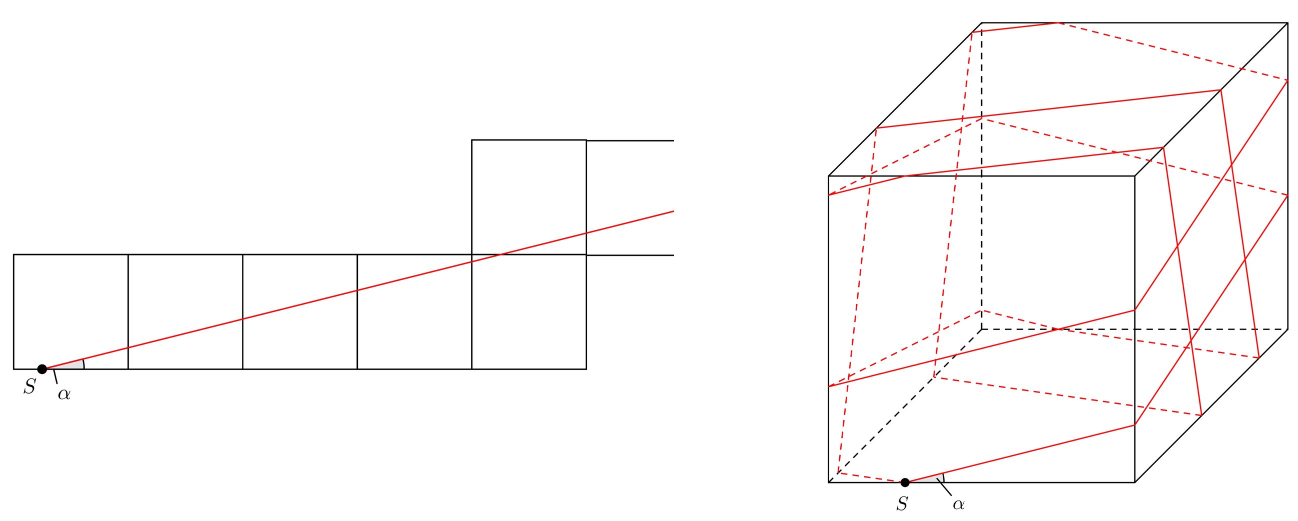


Figure 4

A closed geodesic is called simple if it does not intersect with itself, as we can see in the pictures above. Figure 4 shows a non-simple closed geodesic. Now we can ask the question: Can we classify all simple closed geodesics on the tetrahedron, cube and octahedron?

Let's examine the tetrahedron. We label the four vertices of the tetrahedron with A, B, C and D. Without loss of generality, we say that the geodesic starts on the edge AB and proceeds into the face ABC. Considering the tesselation of the plane by the regular triangle, it turns out that there is only one way to label the vertices of the tesselation with A, B, C and D in such a way, that if you fold every trajectory in the tesselation back to the tetrahedron, the vertices of the tesselation match with the vertices of the tetrahedron. (This doesn't hold for the other platonic solids) (Figure 5)

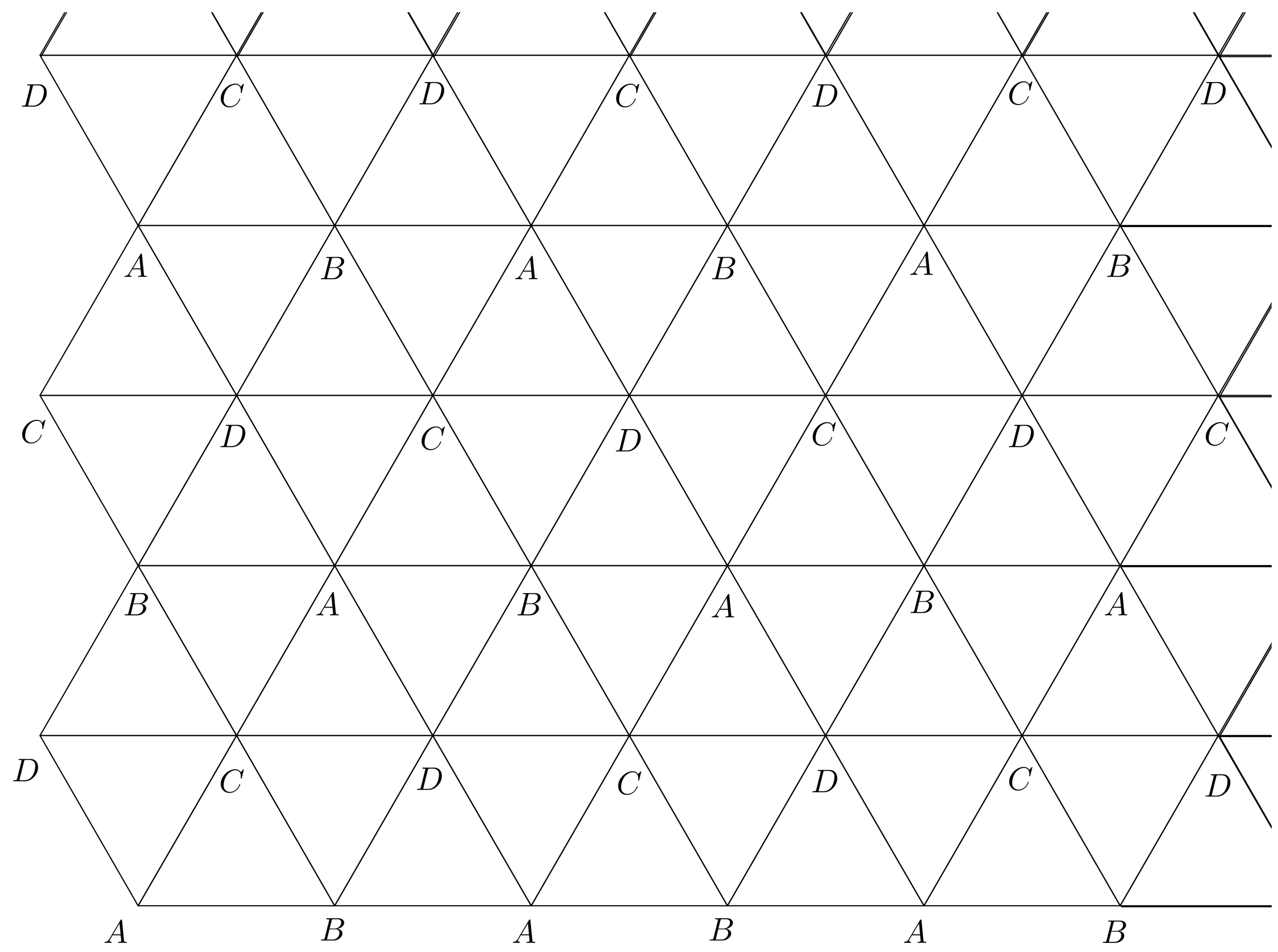


Figure 5

This labeled tesselation (Figure 5) has an interesting property: Every identically labeled triangle is either parallel, or parallel when rotated by π. Therefore it follows that all segments of a geodesic on one face of the tetrahedron are parellel to each other. This implies that every closed geodesic is simple and therefore there exist simple closed geodesics of arbitrarily large length.

On the cube and the octahedron the situation is different. In order to classify all simple closed geodesics Theorem 3 is fundamental.

Theorem 3. *If a closed geodesic on the cube or octahedron intersects one edge at two different points, it follows that the closed geodesic is* ***not*** *simple.*

This is proven by showing algebraically that a certain geometric property applies for all closed geodesics with the property of Theorem 3. Figure 4 depicts an example of Theorem 3.

The cube and the octahedron have 12 edges. Following from Theorem 3, if a closed geodesic has more than 12 segments, it can't be simple, because if it had more than 12 segments, it would interect with an edge at two different points.