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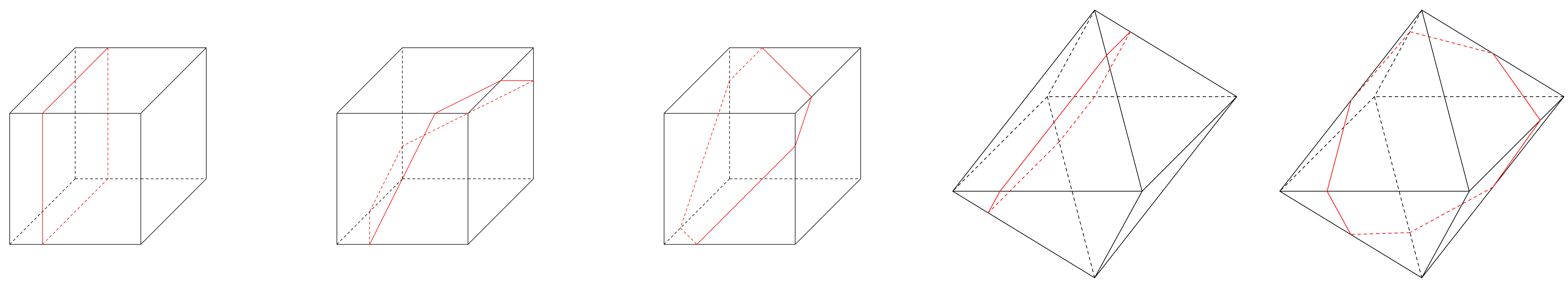
Constantin Kogler, Switzerland

On periodic billiard trajectories in regular polygons and simple closed geodesics on the tetrahedron, cube and octahedron



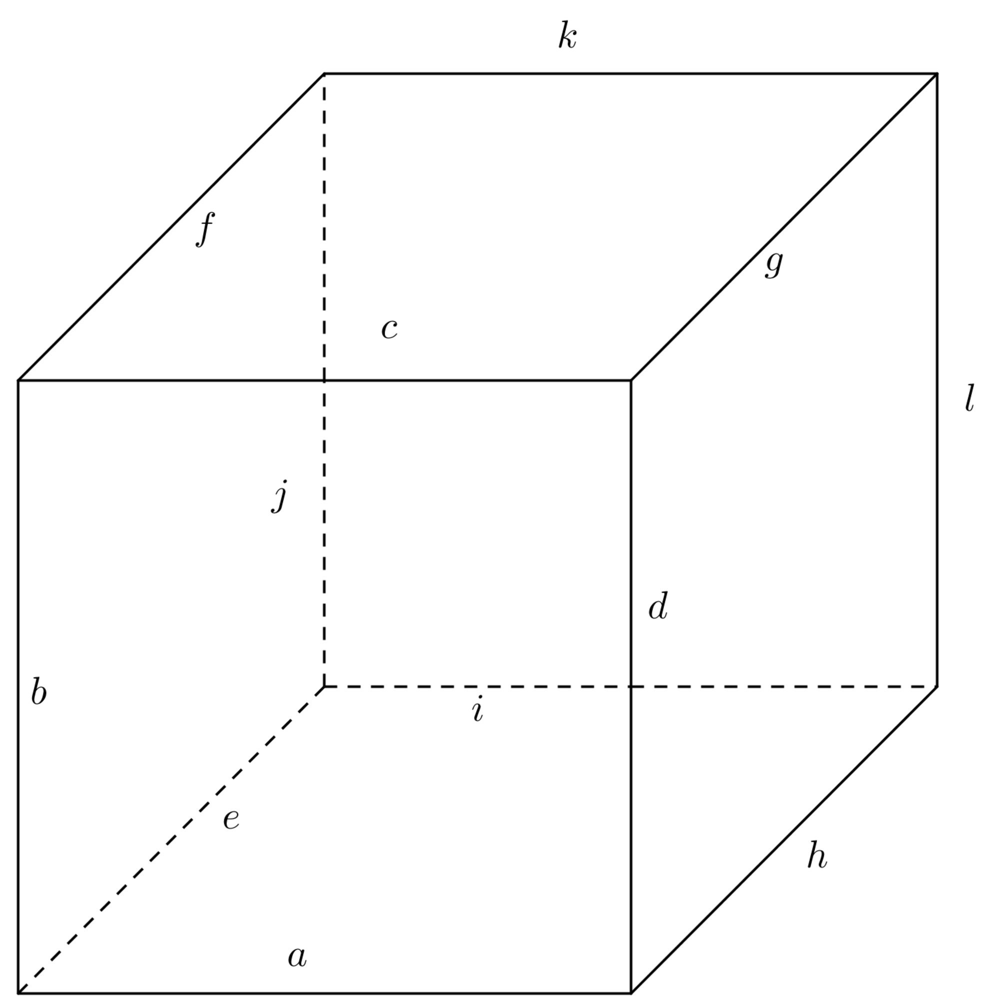


Figure 6: Cube with labeled edges

In order to classify all simple closed geodesics on the cube, we just need to investigate alle closed geodesics with a segment number of 12 or less. The edges of a cube are labeled with $a, b, c, d, e, f, g, h, i, j, k$ and l - as you can see in Figure 6.

The so called letter sequence - consisting of the letters $a, b, c, d, e, f, g, h, i, j, k$ und l - represents all geodesics for which the following property applies: the first intersection with an edge is the first letter of the sequence, the second intersection with an edge is the second letter of the sequence, and so on.

We define that the geodesic starts on edge a - so the letter sequence begins with a - and proceeds into the face of edge a, b, c and d. For reasons of symmetry, we only consider geodesics with a launching angle α with $\alpha \in [0, \frac{\pi}{2}]$. Therefore the only possibilites for the second letter are c and d.

For example, adl represent the set of all geodesics for which the starting point is on the edge a, the first intersection is on the edge d and the second intersection is on the edge l. We can continue the letter sequence with all possible subsequent letters - in this case i,j or k. The three continued letter sequences represent the same geodesics as the preceding one. Some letter sequences definetly don't contain simple closed geodesics. In this way, we can check all possible geodesics by the continuation of the letter sequences which begin with a. The ones which definitely don't contain closed geodesics are eliminated. By this process we find all simple closed geodesics.

Now we continue the letter sequence until one of the following cases appears, for which we don't need to continue the letter sequence:

(1) Contains simple closed geodesic (is marked red)
(2) Not possible according to Theorem 3 (is marked blue)
(3) The net of the geodesic is not possible (is marked green)
(4) Definitely has an intersection (is marked yellow)

Now we begin the continuation of the letter sequence:

Number of letters = 1
a

Number of letters = 2
ac; ad

Number of letters = 3
acf; ack; acg; adg; adl

Number of letters = 4
ackj; acki; ackl; acgh; acgl; adgf; adgk; adli; adlj; adlk

Number of letters = 5
ackie; ackia; ackih; ackld; acklh; acglj; acgli; adgfb; adgfe; adgfj; adgkj; adgki; adlje; adljb; adljf, adlkc, adlkf

Number of letters = 6
ackihd; ackihg; acklha; acklhe; acglje; acgljb; acgljf; acglie; acglia; adgfea; adgfeh; adgfei; adgfji; adgfjl; adgfjk; adgkjf; adgkjb; adgkje; adgkie; adgkia; adgkib; adljba; adljbd; adljbc; adljfg; adljfc; adlkfb; adlkfe

Number of letters = 7
ackihdb; ackihdc; acgljea; acgljeh; acgljbc; acgljbd; acgljba; acglieb; acglief; adgfehd; adgfehg; adgfehl; adgfeil; adgfeik; adgfjia; adgfjih; adgkjbc; adgkjbd; adgkjba; adgkjea; adgkjeh; adgkieb; adgkief; adljbck; adljbcg; adljfgd; adljfca; adljfcd; adlkfbd;adlkfba; adlkfea; adlkfeh; adlkfei

Number of letters = 8
acgliebc; acgliebd; acgliefc; acgliefg; acgliefk; adgfehlk; dgfehlj; adgfeild; adgfeilg; adgfjihd; adgfjihg; adgkiebc; adgkiebd; adljbcgh; adljbcgl

Number of letters = 9
adgfehlkf; adgfehlkc; adgkiebck; adgkiebcg

The letter sequences ackia; acgljea; adgfjia; adgkjea and adlkfea represent simple closed geodesics. Checking each case leads to the result, that on the cube there exist exactly three classes of simple closed geodesics. They have a specific length and lauching angle. The same method also works for the octahedron, with the result that there exist two classes of simple closed geodesics.

References

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